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$N = 2$ strings on orbifolds

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Abstract: We study closed $N = 2$ strings on orbifolds of the form $T^4/\mathbb{Z}_2$ and $\mathbb{C}^2/\mathbb{Z}_2$. We compute the torus partition function and prove its modular invariance. We analyze the BRST cohomology of the theory, construct the vertex operators, and compute three and four point amplitudes of twisted and untwisted states. We introduce a background of D-branes, and compute twist states correlators.

Keywords: Superstrings and Heterotic Strings, D-branes, Extended Supersymmetry.

Contents

1. Introduction

Closed $N = 2$ strings [1, 2] possess local $N = 2$ supersymmetry on the string worldsheet. Critical $N = 2$ strings have a four-dimensional target space [3]. The supersymmetric structure implies that the target space has a complex structure. Therefor it must be of signature $(4, 0)$ or $(2, 2)$. In $(4, 0)$ signature there are no propagating degrees of freedom in the $N = 2$ string spectrum. In $(2, 2)$ signature there are only massless scalars in the spectrum, and the infinite tower of massive excitations of the string is absent.

Closed $N = 2$ strings have been divided to two T-dual families denoted by α and β [4]. β strings are the ones studied in [3], where it was shown that they have self-dual (curvature) four-manifolds as their target space. The α strings effective action has been constructed in [5], where it was shown that the target space dynamics is that of self-dual (curvature) four-manifolds with torsion [6]. D-branes in the $N = 2$ strings have been analyzed in [4, 7]. $N = 2$ strings have global $N = 4$ worldsheet supersymmetry [8]. They have been shown to be equivalent to topological $N = 4$ strings in [9]. They have been suggested as describing the topological sector of six-dimensional little string theories with sixteen supercharges [10].

Despite the simplicity of $N = 2$ strings, very little is known when the target space is curved. Curved backgrounds are of particular importance, when looking for open-closed string duality structure in the $N = 2$ strings. In this paper we will take one step in this direction, and study $N = 2$ strings on orbifolds of the form $T^4/\mathbb{Z}_2$ and $\mathbb{C}^2/\mathbb{Z}_2$. First, we will compute the torus partition function and prove its modular invariance. We will then analyze the BRST cohomology of the theory in different pictures. We will see that the different pictures are not isomorphic. We will find a massless scalar and global degrees of freedom in the untwisted sector, and one twisted state in the twisted sector. We will then construct the vertex operators, compute three and four point amplitudes of twisted and untwisted states, and introduce the low-energy effective action. We will then add a background of D-branes, compute twist states correlators in their background, consider the effective world volume theory and compare it with the unorbifolded case. Finally, we will discuss the results.

The paper is organized as follows. In section 2 we will consider the torus partition function and show that it is modular invariant. We analyze the spectrum and construct the vertex operators. In section 3 we compute three and four point amplitudes of twisted and untwisted states, and interpret the results. In section 4 we will consider closed strings in D-branes background and compute twist states correlators. We discuss the effective world volume theory and compare it with the unorbifolded case.

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2. Partition function and spectrum of the $N = 2$ orbifold

In this section we introduce the torus partition function and show that it is modular invariant. We then introduce the spectrum of the $N = 2$ string on the $\mathbb{Z}_2$ orbifold. We will discuss shortly the spectrum of the untwisted sector, which consists of a massless scalar and global degrees of freedom (graviton, dilaton and a 2-form). We will also perform a BRST analysis in the twisted sector and show the spectrum consists of a single state. We will then construct the corresponding vertex operators.

For an earlier work on the partition function of the $N = 2$ string, see [11]. For an earlier work on BRST analysis in $N = 2$ string orbifolds, see [12, 13]. The interested reader may find detailed derivations of the results in appendices B and C.

2.1 Partition function

The partition function is

$$d^2\tau d\theta \ \mathrm{tr}_\phi \left(\mathcal{P}\, q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) \tag{2.1}$$

where ϕ and $2\pi\theta$ are the periodicities over the α and β cycles of the torus, respectively, of any oscillator with a U(1)-charge 1.

For the $\mathbb{Z}_2$ orbifold, the projection operator is given by $\mathcal{P} = (1 + \mathbf{r})/2$ where $\mathbf{r}$ is the $\mathbb{Z}_2$ reflection.

2.1.1 Oscillators

For the computation of the $N = 2$ string partition function we consider two complex scalars, two holomorphic and two antiholomorphic complex fermions, two sets of holomorphic and antiholomorphic anticommuting ghosts and two sets of holomorphic and antiholomorphic commuting superghosts.

The sum over bosonic oscillators is similar to that in the bosonic string, and can be found in appendix B.2.1.

The fermions give

$$\begin{aligned}
\mathrm{tr}_\phi\left(q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) &= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi\\ -\theta\end{bmatrix}(0,\tau)\right|^4,\\
\mathrm{tr}_\phi\left(\mathbf{r}\cdot q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) &= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi\\ -\theta+1/2\end{bmatrix}(0,\tau)\right|^4.
\end{aligned} \tag{2.2}$$

In the twisted sector

$$\begin{aligned}
\mathrm{tr}_\phi\left(q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) &= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi+1/2\\ -\theta\end{bmatrix}(0,\tau)\right|^4,\\
\mathrm{tr}_\phi\left(\mathbf{r}\cdot q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) &= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi+1/2\\ -\theta+1/2\end{bmatrix}(0,\tau)\right|^4.
\end{aligned} \tag{2.3}$$

For the ghost and superghost sectors one obtains, as in the unorbifolded $N = 2$ string,

$$\mathrm{tr}_\phi\left((-1)^{F_{\mathrm{gh}}}b_0c_0\bar{b}_0\bar{c}_0\,\hat{b}_0\hat{c}_0\bar{\hat{b}}_0\bar{\hat{c}}_0\,q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) = |\eta(\tau)|^{12}\left|\vartheta\begin{bmatrix}\phi\\ -\theta\end{bmatrix}(0,\tau)\right|^{-4}. \tag{2.4}$$

2.2 Modular invariance

Collecting these results together, one obtains

$$Z_{\text{orbifold}} = \frac{1}{2}\int d^2\tau\, d\theta d\phi\, [Z(\tau,\bar{\tau}) + I]\,, \tag{2.5}$$

where $Z(\tau,\bar{\tau})$ is the partition function of the unorbifolded $N = 2$ string,

$$Z(\tau,\bar{\tau}) = \frac{V}{(4\pi^2\alpha'\tau_2)^2}\,, \tag{2.6}$$

in the non-compact case, and

$$Z(\tau,\bar{\tau}) = \sum_{p\in\Gamma} q^{\frac{\alpha'}{4}p_L^2}\bar{q}^{\frac{\alpha'}{4}p_R^2}\,, \tag{2.7}$$

in the compact case, where Γ is the lattice of allowed momenta.

In the non-compact case:

$$I = \frac{|\eta(\tau)|^{12}}{\left|\vartheta\begin{bmatrix}\phi\\-\theta\end{bmatrix}(0,\tau)\right|^4}\left(\frac{\left|\vartheta\begin{bmatrix}\phi\\-\theta+1/2\end{bmatrix}(0,\tau)\right|^4}{|\vartheta_{10}(0,\tau)|^4} + \frac{\left|\vartheta\begin{bmatrix}\phi+1/2\\-\theta\end{bmatrix}(0,\tau)\right|^4}{|\vartheta_{01}(0,\tau)|^4} + \frac{\left|\vartheta\begin{bmatrix}\phi+1/2\\-\theta+1/2\end{bmatrix}(0,\tau)\right|^4}{|\vartheta_{00}(0,\tau)|^4}\right). \tag{2.8}$$

In the compact case I should be multiplied by an overall factor of 2^4.

$Z(\tau,\bar{\tau})$ is known to be modular-invariant after integration by $d\tau d\bar{\tau}$, both for the non-compact case and for the compact case (for the latter this can be shown by Poisson resummation, for an appropriate lattice Γ).

We will now show that I is modular invariant after integration by $d\tau d\bar{\tau} d\theta d\phi$.

Under $\tau \to \tau + 1$, The second and the third terms interchange while the first term remains invariant. This can be shown by taking $\theta \to \theta - \phi - 1/2$ and using

$$\begin{aligned}
\vartheta\begin{bmatrix}a\\b\end{bmatrix}(0,\tau+1) &= e^{-\pi i a(a+1)}\,\vartheta\begin{bmatrix}a\\a+b+1/2\end{bmatrix}(0,\tau)\,,\\
\vartheta\begin{bmatrix}a+1\\b\end{bmatrix}(0,\tau) &= \vartheta\begin{bmatrix}a\\b\end{bmatrix}(0,\tau)\,, \qquad \vartheta\begin{bmatrix}a\\b+1\end{bmatrix}(0,\tau) = e^{2\pi i a}\vartheta\begin{bmatrix}a\\b\end{bmatrix}(0,\tau)\,,
\end{aligned} \tag{2.9}$$

Thus Z_{orbifold} is invariant under $\tau \to \tau + 1$.

Also, one can see that under $\tau \to -1/\tau$ the first and second terms interchange while the third term remains invariant, by taking $\theta \to \phi$ and $\phi \to -\theta$ and using (2.9) and

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}\left(0,-\frac{1}{\tau}\right) = (-i\tau)^{1/2}e^{2\pi i ab}\,\vartheta\begin{bmatrix}b\\-a\end{bmatrix}(0,\tau)\,. \tag{2.10}$$

Thus Z_{orbifold} is invariant under $\tau \to -1/\tau$.

2.3 Spectrum of the $N = 2$ orbifold

We begin by discussing different pictures in the $N = 2$ string. As in the $N = 1$ string, a picture is defined by the superghost sector ground state [14]. In the $N = 2$ string there are two sets of superghosts, and the picture is defined by a pair of numbers rather than a single number. The pictures that will be most relevant for us are $(-1, -1)$, $(0, 0)$ (for NS boundary conditions) and $(-1/2, -1/2)$ (for R boundary conditions), defined in appendix C.

The picture changing operator relates physically equivalent states of different pictures. As has been discussed in [15], the picture changing operator is an isomorphism for non-zero momentum states, but not for zero momentum states. Thus, different pictures may have different states with vanishing momentum.

Another isomorphism between physically equivalent states of different pictures is the spectral flow, which is discussed in section 3 of appendix C.

Because of these isomorphisms, for states with non-vanishing momentum it is sufficient to study the BRST cohomology in the $(-1, -1)$ picture, which consists only of a massless scalar. Thus the spectrum of the unorbifolded $N = 2$ string consists of a massless scalar, and zero momentum states which turn out to describe dilaton, metric and B-field global degrees of freedom [16] (see appendix C.1). The untwisted sector of the $N = 2$ orbifold at hand consists of $\mathbb{Z}_2$ invariant combinations of these states.

BRST analysis of the twisted sector. In the twisted sector there are no momentum degrees of freedom, and the BRST cohomology is different in different pictures:

- In the $(-1, -1)$ picture the normal ordering constant of L_0 is $+1/2$, and The BRST cohomology is empty.

- In the $(-1/2, -1/2)$ picture L_0 has zero normal ordering constant, and the ground state is the only physical state. The corresponding vertex operator is given by

$$c\, e^{-\phi^+/2}\, e^{-\phi^-/2}\,(z)\, \bar{c}\, e^{-\bar{\phi}^+/2}\, e^{-\bar{\phi}^-/2}\,(\bar{z}) \cdot \sigma^1(z, \bar{z})\, \sigma^2(z, \bar{z})\,. \tag{2.11}$$

  where σ^i is the twist field creating the cut in X^i and $\bar{X}^i$ in the z-plane [17], and $\beta^{\mp}\gamma^{\pm} = \partial\phi^{\pm}$ [15].

- In the $(0, 0)$ picture the normal ordering constants of L_0 is $-1/2$, and we get two states in the BRST cohomology, whose matter parts read

$$G^{-\,M}_{-1/2}\psi^1_0\psi^2_0|\sigma\rangle_{(0,0)}\,, \qquad G^{+\,M}_{-1/2}|\sigma\rangle_{(0,0)}\,. \tag{2.12}$$

  These states are physically equivalent to the twisted ground state in the $(-1/2, -1/2)$ picture, through the use of the picture changing operator and spectral flow (see appendix C.3 for details).

3. Scattering amplitudes on $\mathbb{C}^2/\mathbb{Z}_2$ and $T^4/\mathbb{Z}_2$

In this section we will compute scattering amplitudes of untwisted and twisted states in $\mathbb{C}^2/\mathbb{Z}_2$ and $T^4/\mathbb{Z}_2$ backgrounds. We work with $\alpha' = 2$ in this section.

3.1 Four twisted states amplitude

A twisted state is located in target space at a fixed point of the orbifold group. For the $T^4/\mathbb{Z}_2$ orbifold with radius R, they are at $\pi R\epsilon$, where ϵ is a four-dimensional vector (in real coordinates) such that $\epsilon^\mu \in \{0,1\}$. For twisted states located at the fixed points $2\pi f_{\epsilon_i}$, $i=1,\dots,4$, ($f_{\epsilon_i} = \frac{1}{2}R\epsilon_i$) the four twisted states scattering amplitude is

$$\mathcal{A}_{tttt} = \int dx\, d\bar{x} \left\langle c\bar{c}V_{\epsilon_4}^{(-\frac{1}{2},-\frac{1}{2})}(z_\infty)\, c\bar{c}V_{\epsilon_3}^{(-\frac{1}{2},-\frac{1}{2})}(1)\, V_{\epsilon_2}^{(-\frac{1}{2},-\frac{1}{2})}(x,\bar{x})\, c\bar{c}V_{\epsilon_1}^{(-\frac{1}{2},-\frac{1}{2})}(0) \right\rangle_{S^2}, \quad (3.1)$$

where

$$V_{\epsilon_i}^{(-\frac{1}{2},-\frac{1}{2})}(z,\bar{z}) = e^{-\phi^+/2}\, e^{-\phi^-/2}\,(z) \cdot e^{-\bar{\phi}^+/2}\, e^{-\bar{\phi}^-/2}\,(\bar{z}) \cdot \sigma^1_{\epsilon_i}(z,\bar{z})\, \sigma^2_{\epsilon_i}(z,\bar{z})\,. \quad (3.2)$$

$\sigma^j_{\epsilon_i}$ is a twist field for X^j, located at $2\pi f_{\epsilon_i}$.

The matter part of the integrand in (3.1) is

$$\left\langle \sigma^1_{\epsilon_4}\sigma^2_{\epsilon_4}(z_\infty)\, \sigma^1_{\epsilon_3}\sigma^2_{\epsilon_3}(1)\, \sigma^1_{\epsilon_2}\sigma^2_{\epsilon_2}(x,\bar{x})\, \sigma^1_{\epsilon_1}\sigma^2_{\epsilon_1}(0) \right\rangle_{S^2}. \quad (3.3)$$

Define

$$Z(x,\bar{x}) \equiv \lim_{z_\infty\to\infty} |z_\infty| \left\langle \sigma^1_{\epsilon_4}\sigma^2_{\epsilon_4}(z_\infty)\, \sigma^1_{\epsilon_3}\sigma^2_{\epsilon_3}(1)\, \sigma^1_{\epsilon_2}\sigma^2_{\epsilon_2}(x,\bar{x})\, \sigma^1_{\epsilon_1}\sigma^2_{\epsilon_1}(0) \right\rangle_{S^2}. \quad (3.4)$$

Z can be recast as $Z = Z_1 Z_2$, where Z_i denotes a four-point function of twist fields on $T^2/\mathbb{Z}_2$ (or $\mathbb{C}/\mathbb{Z}_2$).

3.1.1 $T^4/\mathbb{Z}_2$ orbifold

Define T^4 as $\mathbb{R}^4/2\pi\Lambda$ with $2\pi\Lambda$ being a four-dimensional lattice. When all the radii of T^4 are equal to R, $\Lambda = \{(m_1R, m_2R, m_3R, m_4R) | m_i \in \mathbb{Z}\}$. For every fixed point $2\pi f_\epsilon$, $2f_\epsilon$ is an element of Λ. Global monodromy conditions (changes of X along closed loops) imply that (3.1) is non-zero only if $\sum f_{\epsilon_i}$ belongs to Λ as well [17].

On $T^4/\mathbb{Z}_2$ one gets [17]

$$Z_i(x,\bar{x}) = \frac{|x(1-x)|^{-1/2}}{\tau_2(x,\bar{x})|F(x)|^2}\, V_{\Lambda_i} \sum_{v_1\in\Lambda_{ic}, v_2\in\Lambda_{ic}'} e^{-\frac{\pi}{4\tau_2}[v_2\bar{v}_2 - \tau_1(v_1\bar{v}_2+\bar{v}_1 v_2)+|\tau|^2 v_1\bar{v}_1]}, \quad (3.5)$$

where $\tau(x) \equiv iF(1-x)/F(x)$, and $F(x)$ is the hypergeometric function

$$F(x) \equiv F\left(\frac{1}{2},\frac{1}{2},1;x\right) = \frac{1}{\pi}\int_0^1 dy\, y^{-1/2}(1-y)^{-1/2}(1-xy)^{-1/2}\,. \quad (3.6)$$

We denoted $\Lambda_c \equiv (1-\theta)(f_{\epsilon_2} - f_{\epsilon_1} + \Lambda)$ and $\Lambda_{c'} \equiv (1-\theta)(f_{\epsilon_2} - f_{\epsilon_3} + \Lambda)$, where θ is the $\mathbb{Z}_2$ rotation. V_Λ is the unit cell volume in Λ. The index i in Λ_{ic}, $\Lambda_{ic'}$, $f^i_{\epsilon_j}$, V_{Λ_i} stands for a projection onto the ith dimension.

The holomorphic ghost part of the four point function reads

$$\begin{aligned}
&\left\langle e^{-\phi^+/2}(z_\infty)e^{-\phi^+/2}(1)e^{-\phi^+/2}(x)e^{-\phi^+/2}(0)\right\rangle_{S^2} \times \\
&\times \left\langle e^{-\phi^-/2}(z_\infty)e^{-\phi^-/2}(1)e^{-\phi^-/2}(x)e^{-\phi^-/2}(0)\right\rangle_{S^2} \times \\
&\qquad\qquad \times \langle c(z_\infty)c(1)c(0)\rangle_{S^2} = {z_\infty}^{1/2} x^{-1/2}(1-x)^{-1/2}\,, \quad (3.7)
\end{aligned}$$

with a similar expression for the anti-holomorphic part. The total four point function is

$$\mathcal{A}_{tttt} = \int dx d\bar{x}\, \frac{|x(1-x)|^{-2}}{\tau_2^2(x,\bar{x})|F(x)|^4} V_\Lambda \sum_{v_1 \in \Lambda_c, v_2 \in \Lambda_{c'}} e^{-\frac{\pi}{4\tau_2}[v_2\bar{v}_2 - \tau_1(v_1\bar{v}_2 + \bar{v}_1 v_2) + |\tau|^2 v_1 \bar{v}_1]}\,, \tag{3.8}$$

where $v_{1,2}$ are two complex dimensional vectors.

It is further shown in appendix D.1 that $\mathcal{A}_{tttt}$ is finite and invariant under crossing symmetry.

Let us discuss this result. The sum over v_1 corresponds to a sum over global monodromy conditions for a contour surrounding 0 and x. In the s-channel ($x \sim 0$), it represents a sum over intermediate winding states (that wind around f_{ϵ_1} and f_{ϵ_2}). The sum over v_2 corresponds to a sum over global monodromy conditions for a contour surrounding x and 1. We can use Poisson resummation and change variables to p, representing the momentum of the intermediate states in the s-channel, as in [17]. Then (3.8) takes the form

$$\mathcal{A}_{tttt} = \int dx d\bar{x} \frac{|x(1-x)|^{-2}}{|F(x)|^4} \sum_{p \in \Lambda^*, v \in \Lambda_c} \exp\left(-2\pi i (f_{\epsilon_2} - f_{\epsilon_3}) \cdot p\right) w^{(p+v/2)^2/2} \bar{w}^{(p-v/2)^2/2}\,, \tag{3.9}$$

where $w(x) \equiv e^{i\pi\tau(x)}$ and Λ^* is the dual lattice of Λ. For instance, when all the T^4 radii are equal, then $\Lambda^* = \{(m_1/R, m_2/R, m_3/R, m_4/R) | m_i \in \mathbb{Z}\}$. Note that the $e^{-2\pi i (f_{\epsilon_2} - f_{\epsilon_3})\cdot p}$ phase is always ± 1. In every term of the sum (3.9), p represents the momentum of the intermediate untwisted state in the s-channel and v/R is its winding number. Thus, the left and right momenta are $p_{L,R} = p \pm v/2$.

$\mathcal{A}_{tttt}$ seems to have poles only at $p_L^2 = p_R^2 = 0$ (see appendix D.2). Performing the integral in (3.9) over a disk of radius a around $x = 0$ gives

$$2\pi \frac{(a/16)^{p_L^2}}{p_L^2} \delta_{p_L^2 - p_R^2} + f(a, p_L^2, p_R^2) \tag{3.10}$$

where f has no poles in p_L^2, p_R^2.

This suggests that in the effective field theory description only poles of intermediate physical states contribute to the scattering amplitude. However, there may be an additional contact term contribution to the amplitude.

3.1.2 $\mathbb{C}^2/\mathbb{Z}_2$

In the non-compact case the global monodromy conditions are trivial, and there is only one fixed point at the origin. We have

$$Z_i(x,\bar{x}) = \frac{2|x(1-x)|^{-1/2}}{F(x)\bar{F}(1-\bar{x}) + F(1-x)\bar{F}(\bar{x})} = \frac{|x(1-x)|^{-1/2}}{\tau_2(x,\bar{x})|F(x)|^2}\,, \tag{3.11}$$

up to a normalization constant of $(2\pi)^{-2}$ which is justified below. The total four point function reads

$$\mathcal{A}_{tttt} = \int dx d\bar{x} Z_1(x,\bar{x}) Z_2(x,\bar{x}) |x(1-x)|^{-1} = (2\pi)^{-4} \int dx d\bar{x} \frac{|x(1-x)|^{-2}}{\tau_2^2(x,\bar{x})|F(x)|^4}\,, \tag{3.12}$$

which is identical to (3.8) for trivial lattices Λ_c, $\Lambda_{c'}$, and is finite.

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Rewriting $1/\tau_2$ as

$$\frac{1}{\tau_2(x,\bar{x})} = \int dp_a dp_b \, e^{-\pi({p_a}^2+{p_b}^2)\tau_2} = \int d^2p \, w^{p^2/2}\bar{w}^{p^2/2} \,, \tag{3.13}$$

we get

$$\mathcal{A}_{tttt} = \int dx d\bar{x} \, \frac{|x(1-x)|^{-2}}{|F(x)|^4} \int \frac{d^4p}{(2\pi)^4} \, w^{p^2/2}\bar{w}^{p^2/2} \,. \tag{3.14}$$

As in the $T^4/\mathbb{Z}_2$ case, this can be interpreted as a sum over intermediate states with poles at $p^2 = 0$. We normalized the amplitude by demanding that for $p = 0$ the leading term in (3.14) at $x \to 0$ is $1 \cdot |x|^{-2}$ [17].

3.2 Twisted-twisted-untwisted states amplitude

The scattering amplitude of two twisted states and one untwisted state, $\mathcal{A}_{utt}$, can be derived using the factorization property of $\mathcal{A}_{tttt}$. In fact, as seen in the previous section, the residue of the pole in (3.10) is equal to $\mathcal{A}_{utt}^2$. Hence

$$\mathcal{A}_{utt} = \sqrt{2\pi} \,. \tag{3.15}$$

Equation (3.15) can also be obtained using the state-operator correspondence. When the twisted states are located on the same fixed point, we have

$$\langle\sigma|\frac{1}{2}(e^{i(p_L\cdot X^L+p_R\cdot X^R)} + e^{-i(p_L\cdot X^L+p_R\cdot X^R)})|\sigma\rangle = 1 \,. \tag{3.16}$$

Here $p \cdot X$ means $p_i\bar{X}^i + \bar{p}_i X^i$. Note that there is no momentum conservation, since there is no target-space translation invariance. This is seen in (3.16) by the absence of the center-of-mass mode x^μ, which would give a delta function in p in the unorbifolded string.

The discussion generalizes to a non-compact target space $\mathbb{C}^2$, by the use of the representation (3.14). We conclude that on-shell, the 3-point function of two twisted states and one untwisted state is independent of momentum, winding number, or the twisted states location, and is equal to one.

In order to construct the effective action involving the twisted state, we need information of the off-shell ttu vertex. Here we assume that it is constant as well. The reason for this assumption will be clearer later on.

3.3 Two twisted and two untwisted states amplitude

We will consider the two twisted - two untwisted states amplitude when the two twisted states are located at the same fixed point. The amplitude is computed as a correlation function of the two untwisted vertex operators, using their Green function in the presence of twist-fields. This method can be used when the total winding number of the two untwisted states is zero.

In the $(0,0)$ picture, the matter part of the untwisted state vertex operator is

$$\frac{1}{2}V^{(0,0)}(k_{L,R}; z, \bar{z}) + \frac{1}{2}V^{(0,0)}(-k_{L,R}; z, \bar{z}) \,, \tag{3.17}$$

where $V^{(0,0)}(k_{L,R};z,\bar{z})$ is given by [5]

$$V^{(0,0)}(k_{L,R};z,\bar{z}) = \left(-k_L\mathcal{J}\partial X + \frac{1}{4}\,(k_L J_+\psi)\,(k_L J_-\psi)\right) e^{ik_L\cdot X^L}\cdot(\text{h.c.})\,. \tag{3.18}$$

$\mathcal{J}$ is the complex structure and $J_\pm \equiv \eta \pm i\mathcal{J}$.

The two twisted - two untwisted states amplitude, for untwisted states with momenta $\pm k_{L,R}$ and $\pm p_{L,R}$, is

$$\begin{aligned}\mathcal{A}_{tt,p,k} = &\int dz d\bar{z}\,\langle\sigma_1(z_\infty)\sigma_2(z_\infty)\,\frac{1}{2}\left(e^{ik_{L,R}\cdot X^{L,R}(1)} + e^{-ik_{L,R}\cdot X^{L,R}(1)}\right)\times\\ &\times\frac{1}{2}\left(V^{(0,0)}(p_{L,R};z,\bar{z}) + V^{(0,0)}(-p_{L,R};z,\bar{z})\right)\,\sigma_1(0)\sigma_2(0)\rangle\,\times \\ &\times\left|\left\langle e^{-\phi^+(z_\infty)/2}e^{-\phi^+(1)}e^{-\phi^+(0)/2}\right\rangle\left\langle e^{-\phi^-(z_\infty)/2}e^{-\phi^-(1)}e^{-\phi^-(0)/2}\right\rangle\langle c(z_\infty)c(1)c(0)\rangle\right|^2,\end{aligned} \tag{3.19}$$

where $k_{L,R}\cdot X^{L,R} \equiv k_L\cdot X^L + k_R\cdot X^R$.

By using the bosonic fields Green function in the presence of two twist fields we get (see appendix E for details)

$$\mathcal{A}_{tt,p,k} = -2\,(p_L\mathcal{J}k_L)\,(p_R\mathcal{J}k_R)\left[I\,(2k_L\cdot p_L, 2k_R\cdot p_R) + I\,(-2k_L\cdot p_L, -2k_R\cdot p_R)\right], \tag{3.20}$$

where

$$\begin{aligned}I(a_L,a_R) &\equiv \int dz\,d\bar{z}\,|z(1-z)^2|^{-1}\left(\frac{1-z^{1/2}}{1+z^{1/2}}\right)^{a_L}\left(\frac{1-\bar{z}^{1/2}}{1+\bar{z}^{1/2}}\right)^{a_R}\\ &= \int_{\mathrm{Im}(v)\geq 0} dvd\bar{v}\,(1-v)^{a_L-1}v^{-a_L-1}(1-\bar{v})^{a_R-1}\bar{v}^{-a_R-1}\,.\end{aligned} \tag{3.21}$$

When $a_L - a_R$ is an integer we may write

$$\begin{aligned}I(a_L,a_R) + I(-a_L,-a_R) &= \int_{\mathbb{C}} dvd\bar{v}\,(1-v)^{a_L-1}v^{-a_L-1}(1-\bar{v})^{a_R-1}\bar{v}^{-a_R-1}\\ &= \pi\frac{\Gamma(-a_L)\Gamma(a_L)\Gamma(1)}{\Gamma(0)\Gamma(1-a_R)\Gamma(1+a_R)} = 0\,.\end{aligned} \tag{3.22}$$

Since $k_L\cdot p_L - k_R\cdot p_R$ must be an integer for the theory to be modular invariant (as in any string theory with a toroidal compactification), we can use (3.22) to calculate the scattering amplitude (3.20). We thus get

$$\mathcal{A}_{tt,p,k} = 0\,. \tag{3.23}$$

All the above is easily generalized to the non-compact background $\mathbb{C}^2/\mathbb{Z}_2$ by setting $p_L = p_R$, $k_L = k_R$.

Study of the function $I(a_L,a_R)$ (see appendix E.1) suggests the following effective field theory description. Only two diagrams contribute to the amplitude, both are of the form *ttu-uuu*, but in one of them the intermediate untwisted state has momentum $\pm(p_{L,R}+k_{L,R})$, and in the other it has momentum $\pm(p_{L,R}-k_{L,R})$. Recall that the momentum of an untwisted state on the orbifold is defined only up to a sign. Only the physical poles (with $(p_{L,R} \pm k_{L,R})^2 = 0$) appear, and there is no two twisted - two untwisted states contact term. The contributions of the two diagrams are $\left[(p\mathcal{J}k)\,(p\mathcal{J}k) - \left((p \pm k)^2\right)^2\right] / (p \pm k)^2$. In fact, the $u(p_1)u(p_2)u(p_3)$ vertex with p_1, p_2 on-shell and p_3 off-shell becomes $(p_1 \mathcal{J} p_2)^2 - (p_3^2)^2$ [3]. Also we have assumed that the off-shell ttu vertex is constant. By noting $(p - k)^2 = -(p + k)^2$, one finds that the two diagrams cancel each other. For the similar reasons, we conjecture that the scattering amplitude of two untwisted states with two twisted states vanishes always, including the cases where the total untwisted states winding number is non-zero, and where the twisted states are located in different fixed points.

Target space picture. We computed three and four-point functions at tree-level of closed $N = 2$ strings on $\mathbb{C}^2/\mathbb{Z}_2$ and $T^4/\mathbb{Z}_2$ orbifolds. These amplitudes can be encoded in a low-energy effective action. It is of the same form as that of the unorbifolded $N = 2$ string [3], with an additional interaction term ttu. We have not ruled out a possible four twisted states contact term $tttt$. In addition to the usual on shell condition $p^2 = 0$ ($p_L^2 = p_R^2 = 0$ for $T^4/\mathbb{Z}_2$), one imposes the condition that every state is $\mathbb{Z}_2$ invariant. Thus, for the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold the on shell untwisted states are of the form $\frac{1}{2}(e^{ipX} + e^{-ipX})$ with $p^2 = 0$.

The low-energy effective action thus takes the form

$$S = \int \left(\partial\phi\bar{\partial}\phi + \frac{1}{3}\,\phi\,\partial\bar{\partial}\phi \wedge \partial\bar{\partial}\phi \, + tt\phi + \mathcal{O}(tttt) \right) \tag{3.24}$$

(ϕ representing an untwisted state and t a twisted one) where the condition that ϕ is $\mathbb{Z}_2$ invariant is imposed on the external states. Also note that the target-space is smaller now (X^i identified with $-X^i$).

3.4 Summary

We have shown the existence of a single twisted state in the $\mathbb{C}^2/\mathbb{Z}_2$ and $T^4/\mathbb{Z}_2$ orbifolds, and computed the low-energy effective action. Unlike the $N = 1$ superstring orbifold, in the $N = 2$ superstring orbifold there is only a single twisted state, so there is only a single (real) modulus. It would be interesting to explore the behavior of this modulus by computing the form of the twist potential term $V(t)$ in the low-energy effective action.

4. D-branes in the $N = 2$ orbifold

The consistent D-branes of closed $N = 2$ strings are of the types $(0+0)$, $(2+0)$, $(0+2)$ and $(2+2)$ branes in $(2,2)$ signature and 0-branes, 2-branes and 4-branes in $(4,0)$ signature (see [7] for details and [18] for an earlier work on D-branes in the unorbifolded $N = 2$ string). In this section we calculate scattering amplitudes of the orbifold theory in the presence of a D-brane located at a fixed point. We then consider the world volume effective theory and compare it with the unorbifolded case. Most of the calculations details which are relevant for this section can be found in appendices F and G.

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4.1 Two twisted states scattering off D-branes

We begin by considering twist fields in open strings. Using this, the XX two-point function in the presence of two twist fields on the disk is found according to its behavior under the conformal symmetries. Finally, the matter part of the two-point function is deduced in a similar method to that used in [17] for the closed bosonic string.

In the closed string the OPEs of the bosonic field with the twist field take the form (omitting index i)

$$\partial X(z)\sigma(w,\bar{w}) \sim (z-w)^{-1/2}\tau(w,\bar{w})\,, \qquad \bar{\partial}X(\bar{z})\sigma(w,\bar{w}) \sim (\bar{z}-\bar{w})^{-1/2}\bar{\tau}(w,\bar{w})\,, \tag{4.1}$$

where τ, $\bar{\tau}$ are excited twist fields. Consider open strings on the upper half plane, Im$z \geq 0$, with the boundary conditions $\partial X(z) = D\bar{\partial}X(\bar{z})$ ($D = \pm 1$) at $z = \bar{z}$. Close to the worldsheet boundary, (4.1) must be corrected by a boundary term. After this correction the OPE takes the form

$$\begin{aligned}\partial X(z)\sigma(w,\bar{w}) &\sim (z-w)^{-1/2}(z-\bar{w})^{-1/2}(w-\bar{w})^{+1/2}\,\tau(w,\bar{w})\,,\\ \bar{\partial}X(\bar{z})\sigma(w,\bar{w}) &\sim D(\bar{z}-\bar{w})^{-1/2}(\bar{z}-w)^{-1/2}(w-\bar{w})^{+1/2}\,\tau(w,\bar{w})\,.\end{aligned} \tag{4.2}$$

For every complex dimension we may thus write

$$\begin{aligned}-\frac{1}{2}\frac{\langle\partial_z X(z)\partial_w \bar{X}(w)\sigma(z_1,\bar{z_1})\sigma(z_2,\bar{z_2})\rangle_{D_2}}{\langle\sigma(z_1,\bar{z_1})\sigma(z_2,\bar{z_2})\rangle_{D_2}} &=\\ = (z-z_1)^{-1/2}(z-z_2)^{-1/2}&(z-\bar{z}_1)^{-1/2}(z-\bar{z}_2)^{-1/2}\times\\ \times(w-z_1)^{-1/2}(w-z_2)^{-1/2}&(w-\bar{z}_1)^{-1/2}(w-\bar{z}_2)^{-1/2}\times g(z,w,z_i,\bar{z}_i)\,,\end{aligned} \tag{4.3}$$

where the function $g(z,w,z_i,\bar{z}_i)$ is single-valued and is holomorphic in z and w. Its form can be deduced from the behavior of (4.3) as $z \to w$ and under the conformal symmetries of the disk.

By subtracting $1/(z-w)^2$ from (4.3) and taking the limit $z \to w$ we get the normalized energy-momentum tensor expectation value

$$\langle T(z)\sigma(z_1,\bar{z_1})\sigma(z_2,\bar{z_2})\rangle_{D_2}/\langle\sigma(z_1,\bar{z_1})\sigma(z_2,\bar{z_2})\rangle_{D_2}\,.$$

In the limit $z \to z_1$, the coefficient of its $(z-z_1)^{-1}$ term is $\partial_{z_1}\ln\langle\sigma(z_1,\bar{z_1})\sigma(z_2,\bar{z_2})\rangle_{D_2}$. Repeating this procedure for the antiholomorphic part of the energy-momentum tensor and integrating, gives the twist fields two-point function on the disk. we get

$$\langle\sigma(z_1,\bar{z_1})\sigma(z_2,\bar{z_2})\rangle_{D_2} = (z_1-\bar{z_1})^{A+B-\frac{1}{12}}(z_1-z_2)^{-B-\frac{1}{12}}(z_1-\bar{z_2})^{-A-\frac{1}{12}}\times(\text{c.c.})\,. \tag{4.4}$$

Where A, B are yet unknown constants.

Looking at the four twisted states correlation function on S^2 (3.9), we see that the OPE of two twist fields is of the form (ignoring possible boundary terms):

$$\sigma(z_1,\bar{z_1})\sigma(z_2,\bar{z_2}) \sim (z_1-z_2)^{-1/4}(\bar{z}_1-\bar{z}_2)^{-1/4}\cdots \tag{4.5}$$

Therefore in (4.4) one must have $B = 1/6$.

The two twisted states correlation function in the presence of a D-brane is

$$\begin{aligned}\mathcal{A}_{ttD} = & \int dz_1 \, \langle \sigma^1\sigma^2(z_1,\bar{z}_1)\sigma^1\sigma^2(z_2,\bar{z}_2)\rangle_{D_2} \times \\ & \times \langle e^{-\phi^+/2}(z_1)e^{-\phi^+/2}(z_2)e^{-\bar{\phi}^+/2}(\bar{z}_1)e^{-\bar{\phi}^+/2}(\bar{z}_2)\rangle_{D_2} \times \\ & \times \langle e^{-\phi^-/2}(z_1)e^{-\phi^-/2}(z_2)e^{-\bar{\phi}^-/2}(\bar{z}_1)e^{-\bar{\phi}^-/2}(\bar{z}_2)\rangle_{D_2} \cdot \langle c(z_2)\bar{c}(\bar{z}_1)\bar{c}(\bar{z}_2)\rangle_{D_2} \\ = & \int dz_1 \, (z_1-\bar{z}_1)^{4A-\frac{1}{6}}(z_1-z_2)^{-1}(z_1-\bar{z}_2)^{-2A-\frac{2}{3}}(\bar{z}_1-z_2)^{-2A+\frac{1}{3}}(z_2-\bar{z}_2)^{\frac{1}{2}} \, . \end{aligned} \tag{4.6}$$

This should be invariant under the coordinate transformation $z \to -1/z$, which gives $A = -\frac{1}{12}$. In fact, this is also necessary in order to ensure that (4.4) is invariant under the exchange of the twist fields $(z_1, \bar{z}_1) \leftrightarrow (z_2, \bar{z}_2)$. By substituting $A = -\frac{1}{12}$ into (4.4) we see that the open string OPE of two twist fields near the worldsheet boundary includes no boundary terms. On the disk we may fix $z_2 = i$, $z_1 = iy$ where y is a real number between 0 and 1. Thus,

$$\mathcal{A}_{ttD} = \int_0^1 dy \, y^{-1/2}(1-y)^{-1} \, . \tag{4.7}$$

The integral diverges. The divergence comes from the area of integration $y \to 1$, where the two twist fields are close to each other. Note that the leading term of the integrand at this regime is $(1-y)^{-1}$. This should be expected, because the OPE of the two twisted states located at iy and i should be of the form $(1-y)^{-1+p^2}$ where p is the momentum of the intermediate state; $p = 0$ contributes the leading term.

We propose the following low-energy effective action picture: The diagram contributing to $\mathcal{A}_{ttD}$ is a combination of a ttu diagram, and a coupling term of u to the D-brane, with a propagator uu connecting them. The coupling of a single u to a D-brane is computed in [7] and found to be given by a constant. Recall also that the off-shell ttu vertex assumed to be constant as well. Hence the diagram is proportional to $\int d^{4-p}k_\perp \, 1/k_\perp^2$, where $k_\perp$ is the momentum of the intermediate untwisted state normal to the Dp-brane world volume. This is because both the ttu vertex and the D-brane break translation invariance along the normal directions of the D-brane. We now interpret that this divergent integral is responsible for the divergent result of $\mathcal{A}_{ttD}$.

Other amplitudes. Scattering amplitudes of untwisted states in the presence of D-branes are identical to those found in [7],[1] with the appropriate $\mathbb{Z}_2$ invariance taken into account. Thus, instead of each string of momentum p, one sums up the amplitudes for a string of momentum p and the amplitudes for a string of momentum $-p$. We find that all the amplitudes are unchanged. The only exception is the three open string untwisted states amplitude, which is zero in the orbifold theory.

4.2 Effective world volume theory

The $(2+2)$ **brane.** In the unorbifolded theory, the effective theory in the case of the $(2+2)$ (space filling) brane is self-dual Yang-Mills coupled to gravity [19]. The self-dual

[1]Some of these amplitudes were previously computed in [18].

YM equation reads

$$F_{IJ} = \frac{1}{2}\epsilon_{IJKL}F^{KL}\,, \tag{4.8}$$

where

$$F_{IJ} = -[D_I, D_J], \quad D_I = \partial_I - A_I\,. \tag{4.9}$$

and (in complex coordinates)

$$A_i \equiv e^{-i\varphi/2}\partial_i e^{i\varphi/2}\ ,\ \bar{A}_{\bar{i}} \equiv e^{i\varphi/2}\bar{\partial}_{\bar{i}} e^{-i\varphi/2} \tag{4.10}$$

with φ being the open string state.

In the orbifold theory, the only difference is that now the closed string state is coupled to the twisted state. Thus, the effective action should be the same as in [19] but with the additional ϕtt term (and possible terms of order t^4). It reads

$$S = \int \left(\partial\phi\bar{\partial}\phi + \frac{1}{3}\phi\partial\bar{\partial}\phi \wedge \partial\bar{\partial}\phi + tt\phi + \mathcal{O}(t^4) - \mathrm{Tr}(\partial\varphi\bar{\partial}\varphi) + \partial\bar{\partial}\phi \wedge \mathrm{Tr}(\varphi\partial\bar{\partial}\varphi) + \mathcal{O}(\varphi^3)\right) \tag{4.11}$$

where $\mathbb{Z}_2$ acts on both the target space indices and the Chan-Paton indices, yielding the quiver-type gauge theories [20].

Lower dimensional branes. In lower dimensional branes the unorbifolded world volume theory is a dimensional reduction of self-dual Yang-Mills [7]. However, due to (4.10) A_i vanish in all directions perpendicular to the brane, so that there are no worldvolume fields that correspond to these directions. For the $(2+0)$-brane the equations of motion now read [19]

$$\bar{\partial}\left(e^{-i\varphi/2}\partial e^{i\varphi/2}\right) = 0\,, \qquad \partial\left(e^{i\varphi/2}\bar{\partial} e^{-i\varphi/2}\right) = 0\,, \tag{4.12}$$

giving a WZW theory. Closed - open strings scatterring amplitudes vanish, while closed string one-point function on D_2 is non-vanishing [7]. Thus, for the (2+0)-brane the effective world-volume theory is a two-dimensional WZW theory with only one coupling to the bulk given by the one-point function.

In the orbifolded theory, we have seen no coupling of open states to twisted states up to the level considered (namely, no twisted-twisted-open pole appeared in $\mathcal{A}_{ttD}$). Therefor the effective world-volume theory is again a two-dimensional WZW theory, the only difference being the target-space identification under $\mathbb{Z}_2$ (which changes the world-volume topology as well).

In the $(0+0)$-brane there are no degrees of freedom as A_i vanishes in all directions, and the world-volume theory is trivial.

4.3 Summary

We have computed twist correlators in the presence of D-branes and discussed the world volume effective action. The twisted states exhibit infinite scattering amplitudes off D-branes. We leave the further investigation of this to future work.

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A. Free fields representation of the $N = 2$ superconformal algebra

The $N = 2$ superconformal algebra is

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n} \\
[L_n, J_m] &= -mJ_{m+n} \\
\left[L_n, G_r^{\pm}\right] &= \left(\frac{n}{2} - r\right) G^{\pm}_{n+r} \\
\left[J_n, G_r^{\pm}\right] &= \pm G^{\pm}_{n+r} \\
\{G_r^{\pm}, G_s^{\pm}\} &= 0 \\
\{G_r^{+}, G_s^{-}\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}\,.
\end{aligned} \tag{A.1}
$$

The free fields representation of this algebra reads

$$
\begin{aligned}
L_n^M &= \frac{1}{2}\alpha_k^i\bar{\alpha}_{i\ n-k} + \frac{1}{4}(2t-n)\bar{\psi}^i_{n-t}\psi_{i\ t} + a^M\delta_n\,, \\
L_n^g &= (2n-m)b_m c_{n-m} + (n-m)\hat{b}_m\hat{c}_{n-m} + \left(\frac{3}{2}n - s\right)\beta_s^+\gamma^-_{n-s} + \\
&\quad + \left(\frac{3}{2}n - s\right)\beta_s^-\gamma^+_{n-s} + a^g\delta_n\,, \\
G_r^{M+} &= \frac{1}{\sqrt{2}}\bar{\alpha}_k^i\psi_{i\ r-k}\,, \qquad G_r^{M-} = \frac{1}{\sqrt{2}}\alpha_k^i\bar{\psi}_{i\ r-k} \\
G_r^{g\pm} &= -\frac{1}{2}(2r+n)\beta^{\pm}_{r-n}c_n \pm \hat{c}_n\beta^{\pm}_{r-n} - 2b_n\gamma^{\pm}_{r-n} \mp (2r-n)\hat{b}_n\gamma^{\pm}_{r-n}\,, \\
J_n^M &= \frac{1}{2}\psi_t^i\bar{\psi}_{i\ n-t} + a_J^M\delta_n\,, \\
J_n^g &= \gamma_r^-\beta^+_{n-r} - \gamma_r^+\beta^-_{n-r} - nc_m\hat{b}_{n-m} + a_J^g\delta_n\,.
\end{aligned} \tag{A.2}
$$

where $n, m \in \mathbb{Z}$, $k \in \mathbb{Z} + \phi_0$, $t \in \mathbb{Z} + \frac{1}{2} + \phi + \phi_0$, $r, s \in \mathbb{Z} + \frac{1}{2} \pm \phi$.

And the BRST charge is

$$
\begin{aligned}
Q_{\mathrm{BRST}} = &: \left(L_n^M + \frac{1}{2}L_n^g + \frac{a^g}{2}\right)c_{-n} : + : \left(J_n^M + \frac{1}{2}J_n^g + \frac{a_J^g}{2}\right)\hat{c}_{-n} : \\
&+ : \left(G_r^{M+} + \frac{1}{2}G_r^{g+}\right)\gamma^-_{-r} : + : \left(G_r^{M-} + \frac{1}{2}G_r^{g-}\right)\gamma^+_{-r} :
\end{aligned} \tag{A.3}
$$

So that

$$
\begin{aligned}
L_n &= \{Q_{\mathrm{BRST}}, b_n\} \\
J_n &= \{Q_{\mathrm{BRST}}, \hat{b}_n\} \\
G_r^{\pm} &= [Q_{\mathrm{BRST}}, \beta^{\pm}]
\end{aligned} \tag{A.4}
$$

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B. Partition function

B.1 Moduli integration measure

B.1.1 The $\mathrm{U}(1)_V \times \mathrm{U}(1)_A$ gauge transformations

A Polyakov-type worldsheet description of the $N = 2$ string is given by the action of two-dimensional $(2, 2)$ supergravity. The fields are a graviton, two gravitini and a vector field, interacting with complex scalars X^i, and Dirac spinor fermions $\Psi^i = (\psi_L^i, \psi_R^i)$, with $i = 1, 2$. The gravitini can be set to zero by the supersymmetry transformations, and in the following we will consider vector field configurations where the worldsheet instanton number is zero (see e.g. [21] for instanton numbers different than zero). We will be working with euclidean worldsheets. The two-dimensional supergravity action reads then

$$S = \frac{1}{2\pi}\int d^2\sigma g^{1/2}\left(\frac{1}{2\alpha'}\partial_a X^i \partial^a \bar{X}_i + \frac{i}{2}{\Psi^i}^\dagger \Gamma^a \partial_a \Psi_i + A_a {\Psi^i}^\dagger \Gamma^a \Psi_i\right), \tag{B.1}$$

where $a = 1, 2$, A_a is a real vector field, $\Gamma^a = \sigma^a$ and

$${\psi_L^i}^\dagger = \bar{\psi}_R^i\,, \qquad {\psi_R^i}^\dagger = \bar{\psi}_L^i\,. \tag{B.2}$$

The index i in X^i, $\psi_{L,R}^i$ ($\bar{X}^i$, $\bar{\psi}_{L,R}^i$) is understood as a holomorphic (anti-holomorphic) index of the target space.

It was pointed out in [22] (in the lorentzian case) that the two-dimensional supergravity action has both axial and vector gauge symmetries, and that they are both required in the gauge fixing procedure. Consider the vector and axial gauge symmetries, whose transformations are parameterized by the real parameters α_V and α_A,

$$A_a \to A_a + \frac{1}{2}(\partial_a \alpha_V + \epsilon_{ab}\partial^b \alpha_A)\,, \qquad \Psi^i \to e^{i\alpha_V + \sigma^3 \alpha_A}\Psi^i\,, \tag{B.3}$$

$\epsilon_{12} = 1$. Defining $w = \sigma_1 + i\sigma_2$,

$$\begin{aligned}
A \equiv A^w &\to A^w + \bar{\partial}\alpha\,, & \bar{A} \equiv A^{\bar{w}} &\to A^{\bar{w}} + \partial\bar{\alpha} \\
\psi_L^i &\to e^{i\alpha}\psi_L^i\,, & \psi_R^i &\to e^{i\bar{\alpha}}\psi_R^i \\
\bar{\psi}_L^i &\to e^{-i\alpha}\bar{\psi}_L^i\,, & \bar{\psi}_R^i &\to e^{-i\bar{\alpha}}\bar{\psi}_R^i\,,
\end{aligned} \tag{B.4}$$

where $\alpha \equiv \alpha_V - i\alpha_A$. Note that $\bar{A} = A^\dagger$.

The generalization of (B.4) for a field $\mathcal{O}$ with U(1) charges J_0, $\bar{J}_0$ with respect to the gauge fields A, $\bar{A}$ reads

$$\mathcal{O} \to e^{i(\alpha J_0 + \bar{\alpha}\bar{J}_0)}\mathcal{O}\,. \tag{B.5}$$

Here $\psi_{L,R}^i$ may be periodic up to the phases $2\pi\phi_{L,R}$, which is more general than the allowed periodicity of the $N = 1$ superstring. In this case $\bar{\psi}_{L,R}^i$ are periodic up to the phases $-2\pi\phi_{L,R}$. However, (B.2) implies $\phi' \equiv \phi_L = \phi_R$. This is consistent with the $N = 2$ algebra only if the supercurrents, the superghosts and anti-superghosts have the same periodicity as the fermions of the same U(1) charges. Thus, every field $\mathcal{O}$ satisfies

$$\mathcal{O}(w + 2\pi, \bar{w} + 2\pi) = e^{2\pi i(J_0 + \bar{J}_0)\phi'}\mathcal{O}(w, \bar{w})\,, \tag{B.6}$$

with the same ϕ' for all fields.

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B.1.2 Moduli on T^2 and their measure

The vector field A is a two-component field, which may be set to zero by the two gauge transformations. However, we must consider possible Killing vectors, i.e. gauge transformations which do not change A, and moduli which are variations of A orthogonal to the gauge transformations.

The Killing vectors are found by solving the equation

$$\partial_a \alpha_V + \epsilon_{ab} \partial^b \alpha_A = 0\,, \tag{B.7}$$

or equivalently

$$\partial \alpha = 0\,. \tag{B.8}$$

On a compact Riemann surface, the only solution for this equation is $\alpha =$ const., and for all closed topologies, there is one complex (or two real) Killing vectors. Therefor, there will be one insertion of $\hat{c}$ (the U(1) ghost), and one insertion of $\bar{\hat{c}}$ in every amplitude. Since the Killing vectors are constant, these insertions do not affect the calculations.

The moduli are found by solving:

$$\int d^2\sigma\, g^{1/2} \delta A^a (\partial_a \alpha_V + \epsilon_{ab} \partial^b \alpha_A) = 0\,, \tag{B.9}$$

or similarly

$$\int d^2\sigma\, g^{1/2} (\nabla_a (\delta A^a) \alpha_V + \epsilon_{ab} \nabla^b (\delta A^a) \alpha_A) = 0\,. \tag{B.10}$$

Thus, we have $\nabla_a(\delta A^a) = 0$, $\epsilon_{ab}\nabla^b(\delta A^a) = 0$ or equivalently

$$\partial(\delta A) = 0\,, \qquad \bar{\partial}(\delta \bar{A}) = 0\,. \tag{B.11}$$

These equations have no solution on S^2, and therefor $A, \bar{A}$ can be set to zero on the 2-sphere.

On T^2, A can be gauge transformed to a complex constant. This can be seen directly by expanding A, which is a doubly periodic function, in a sine and cosine series and constructing explicitly the gauge transformation function.[2] Alternatively, a solution to (B.11) is given by a holomorphic one form of T^2 is thus a (complex) constant. We will therefor have one complex modulus. The path integral on the torus worldsheet includes a sum over all configurations of the gauge fields, which are the constants after gauge fixing.

The fermions obey the equations of motion:

$$\bar{\partial}\psi_L = iA\psi_L\,, \qquad \partial\psi_R = i\bar{A}\psi_R\,, \tag{B.12}$$

from which it follows that

$$\psi_L(w,\bar{w}) = e^{i(A\bar{w}+\bar{A}w)}\psi'_L(w)\,, \qquad \psi_R(w,\bar{w}) = e^{i(A\bar{w}+\bar{A}w)}\psi'_R(\bar{w})\,, \tag{B.13}$$

[2] A will be expanded in $\cos(a_{m,n}w + \bar{a}_{m,n}\bar{w})$ and $\sin(a_{m,n}w + \bar{a}_{m,n}\bar{w})$ where $a_{m,n} \equiv \frac{i}{2\tau_2}(m\bar{\tau} + n)$.

where $\psi'_L(w)$ is holomorphic and $\psi'_R(\bar{w})$ is antiholomorphic. Since $\psi_{L,R}(w,\bar{w})$ are truly doubly periodic, the fields $\psi'_{L,R}$ are doubly periodic only up to the phase shifts

$$\psi'_L(w+2\pi\tau) = e^{-2i\pi(A\bar{\tau}+\bar{A}\tau)}\psi'_L(w)\,, \qquad \psi'_L(w+2\pi) = e^{-2i\pi(A+\bar{A})}\psi'_L(w)\,, \tag{B.14}$$

with the same phase shifts for ψ'_R as well. We denote the phase shifts by

$$\theta = -(\bar{\tau}A+\tau\bar{A})\,, \qquad \phi = -(A+\bar{A})\,. \tag{B.15}$$

It will later be useful to define $u \equiv -\theta+\phi\tau$, so that $A = i(\phi\tau-\theta)/2\tau_2 = iu/2\tau_2$.

The same holds for any field with a non-zero U(1) charge (including the superghosts). They can be written as a holomorphic (antiholomorphic) field multiplied by a Wilson line, with the holomorphic (antiholomorphic) field being doubly periodic only up to phase shifts

$$\mathcal{O}(w+2\pi\tau,\bar{w}+2\pi\bar{\tau}) = e^{2\pi i(J_0+\bar{J}_0)\theta}\mathcal{O}(w,\bar{w})\,, \qquad \mathcal{O}(w+2\pi,\bar{w}+2\pi) = e^{2\pi i(J_0+\bar{J}_0)\phi}\mathcal{O}(w,\bar{w})\,. \tag{B.16}$$

Note that any phase shift ϕ' as in (B.6) can be absorbed in ϕ by a redefinition of the gauge field.

The partition function on the torus is defined by tracing over all oscillators on the unit circle (the α-cycle) shifted by $2\pi\tau$, with the non-trivial periodicities (B.16). That along the α-cycle (parametrized by the phase shift $2\pi\phi$) affects the conformal weight of these operators. One has to insert the operator $e^{-2\pi i\theta(J_0+\bar{J}_0)}$ into the trace in order to impose the boundary condition in (B.16) along the β-cycle.

Another approach is to add $\frac{1}{2\pi}\int d^2\sigma g^{1/2}[-2(AJ+\bar{A}\bar{J})] = 2\pi(uJ_0+\bar{u}\bar{J}_0)$ to the action. These two approaches are equivalent, as is implied by the equality (to be confirmed later)

$$d^2\tau d\theta\ \mathrm{tr}_\phi\left(q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) = d^2\tau d\theta d\phi\ \mathrm{tr}\left(q^{L_0}\bar{q}^{\bar{L}_0}e^{2\pi i(uJ_0+\bar{u}\bar{J}_0)}\right). \tag{B.17}$$

Finally we should compute the integration measure. Tracing over ϕ and θ replaces the tracing over the gauge field. There is one complex modulus u for the gauge field A. We get

$$\partial_u A = \frac{i}{2\tau_2}\,, \qquad \partial_{\bar{u}}\bar{A} = -\frac{i}{2\tau_2}\,. \tag{B.18}$$

The integration measures can now be calculated from the ghost and antighost insertions

$$\begin{aligned}\frac{1}{4\pi}\left(\hat{b},\partial_u A\right)\hat{c} &= \frac{i}{8\pi\tau_2}\int d^2\sigma g^{1/2}\hat{b}\hat{c} = \frac{\pi}{2}\hat{b}_0\hat{c}_0\,,\\ \frac{1}{4\pi}\left(\bar{\hat{b}},\partial_{\bar{u}}\bar{A}\right)\bar{\hat{c}} &= \frac{-i}{8\pi\tau_2}\int d^2\sigma g^{1/2}\bar{\hat{b}}\bar{\hat{c}} = \frac{\pi}{2}\bar{\hat{b}}_0\bar{\hat{c}}_0\,.\end{aligned} \tag{B.19}$$

Note that the torus amplitude contains two U(1) anti-ghost insertions for the one complex modulus. The total integration measure for all moduli reads

$$\frac{d^2\tau}{4\tau_2}dud\bar{u}\frac{1}{4\pi}\left(\hat{b},\partial_u A\right)\hat{c}\frac{1}{4\pi}\left(\bar{\hat{b}},\partial_{\bar{u}}\bar{A}\right)\bar{\hat{c}} = \frac{\pi^2}{8}d^2\tau d\theta d\phi\hat{b}_0\hat{c}_0\bar{\hat{b}}_0\bar{\hat{c}}_0\,. \tag{B.20}$$

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B.2 Oscillators

B.2.1 Bosons

Working with the complex coordinates z, $\bar{z}$ with $z = e^{-iw}$, we may Laurent expand the complex scalars:

$$\partial X^i(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_r \frac{\alpha^i_r}{z^{r+1}}\,, \qquad \partial \bar{X}^i(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_s \frac{\bar{\alpha}^i_s}{z^{s+1}}\,. \tag{B.21}$$

Here

$$r \in \mathbb{Z} + \phi_0\,, \qquad s \in \mathbb{Z} - \phi_0\,, \tag{B.22}$$

where $0 \leq \phi_0 < 1$ gives a twist. For a $\mathbb{Z}_N$ twist, $\phi_0 = k/N$, $k = 0, 1, \ldots, N-1$. Similar expansions in $\bar{z}$ hold for $\bar{\partial} X^i$ and $\bar{\partial}\bar{X}^i$, but with $\phi_0 \to -\phi_0$ (this is due to the fact that as $w \to w + 2\pi$, $z \to e^{-2\pi i} z$ while $\bar{z} \to e^{2\pi i}\bar{z}$).

The commutation relations read

$$[\alpha^i_r, \bar{\alpha}^j_s] = 2r\,\delta_{r+s}\,\eta^{ij}\,, \tag{B.23}$$

and the Virasoro generators take the form

$$L_n = \frac{1}{2} \sum_{r \in \mathbb{Z}+\phi_0} :\alpha^i_r \bar{\alpha}_{i\,n-r}: + a\,\delta_n\,. \tag{B.24}$$

The ground state $|0\rangle_X$ is defined by

$$\alpha^i_r |0\rangle_X = \bar{\alpha}^i_r |0\rangle_X = 0\,, \qquad \text{for } r > 0\,. \tag{B.25}$$

Note that for $\phi_0 = 0$, there is a set of states $|0;p\rangle_X$ defined by their (complex) eigenvalues under α^i_0 (which are the complex conjugates of the eigenvalues under $\bar{\alpha}^i_0$). We denote these eigenvalues by $\sqrt{\frac{\alpha'}{2}} p^i_L$ and $\sqrt{\frac{\alpha'}{2}} p^i_R$ for the holomorphic and anti-holomorphic oscillators, respectively.

For every complex scalar, a can be evaluated in the w coordinates using a zeta function regularization:

$$a = \frac{1}{2} \sum_{n=0,1,\ldots} (n + \phi_0) + \frac{1}{2} \sum_{n=1,2,\ldots} (n - \phi_0) = -\frac{1}{12} + \frac{1}{2}\phi_0(1-\phi_0)\,. \tag{B.26}$$

For the computation of the $N = 2$ string partition function we consider two complex scalars, and each is expanded in both holomorphic modes and anti-holomorphic modes. For the $\mathbb{Z}_2$ orbifold, the projection operator is given by $\mathcal{P} = (1 + \mathbf{r})/2$ where $\mathbf{r}$ is the $\mathbb{Z}_2$ reflection. In the untwisted sector $\phi_0 = 0$, and one finds that in the compact case $T^4/\mathbb{Z}_2$,

$$\begin{aligned} \mathrm{tr}_\phi\left(q^{L_0} \bar{q}^{\bar{L}_0} e^{-2\pi i\theta(J_0 + \bar{J}_0)}\right) &= \sum_{p \in \Gamma} \langle p|p\rangle\, q^{\frac{\alpha'}{4}p_L^2 - \frac{1}{6}} \bar{q}^{\frac{\alpha'}{4}p_R^2 - \frac{1}{6}} \prod_{n=1}^{\infty} |1 - q^n|^{-8} \\ &= \sum_{p \in \Gamma} q^{\frac{\alpha'}{4}p_L^2} \bar{q}^{\frac{\alpha'}{4}p_R^2} |\eta(\tau)|^{-8}\,, \end{aligned}$$

$$\mathrm{tr}_\phi\left(\mathbf{r}\cdot q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) = \sum_{p\in\Gamma}\langle -p|p\rangle\, q^{\frac{\alpha'}{4}p_L^2-\frac{1}{6}}\bar{q}^{\frac{\alpha'}{4}p_R^2-\frac{1}{6}}\prod_{n=1}^{\infty}|1+q^n|^{-8}$$
$$= 2^4\cdot\frac{|\eta(\tau)|^4}{|\vartheta_{10}(0,\tau)|^4}\,. \qquad \text{(B.27)}$$

Where $p\equiv(p_L,p_R)$ and Γ is an 8-dimensional lattice which is self-dual and even under the product $p\cdot p'\equiv p_L\cdot p'_L - p_R\cdot p'_R$ (this is needed for modular invariance).

In the non-compact case $\mathbb{C}^2/\mathbb{Z}_2$, $p\equiv p_L=p_R$ and the sum is replaced by an integral over p. $\langle -p|p\rangle=\delta^4(p+p)=2^{-4}\delta(p)$, $\langle p|p\rangle=\delta^4(0)=(2\pi)^{-4}V$, and we get

$$\mathrm{tr}_\phi\left(q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) = \int d^4p\langle p|p\rangle(q\bar{q})^{\frac{\alpha'}{4}p^2-\frac{1}{6}}\prod_{n=1}^{\infty}|1-q^n|^{-8} = \frac{V}{(4\pi^2\alpha'\tau_2)^2}|\eta(\tau)|^{-8},$$
$$\mathrm{tr}_\phi\left(\mathbf{r}\cdot q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) = \int d^4p\langle -p|p\rangle(q\bar{q})^{\frac{\alpha'}{4}p^2-\frac{1}{6}}\prod_{n=1}^{\infty}|1+q^n|^{-8} = \frac{|\eta(\tau)|^4}{|\vartheta_{10}(0,\tau)|^4}. \qquad \text{(B.28)}$$

For each twisted sector ($\phi_0=1/2$), one gets

$$\mathrm{tr}_\phi\left(q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) = (q\bar{q})^{\frac{1}{12}}\prod_{n=1}^{\infty}\left|1-q^{n-\frac{1}{2}}\right|^{-8} = \frac{|\eta(\tau)|^4}{|\vartheta_{01}(0,\tau)|^4},$$
$$\mathrm{tr}_\phi\left(\mathbf{r}\cdot q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) = (q\bar{q})^{\frac{1}{12}}\prod_{n=1}^{\infty}\left|1+q^{n-\frac{1}{2}}\right|^{-8} = \frac{|\eta(\tau)|^4}{|\vartheta_{00}(0,\tau)|^4}\,. \qquad \text{(B.29)}$$

In a compact case there are 2^4 fixed points, and therefor 2^4 twisted sectors. In the non-compact case there is only one twisted sector.

B.2.2 Fermions

Each holomorphic complex fermion can be Laurent expanded as follows

$$\psi_L^i(z)=\sum_r\frac{\psi_r^i}{z^{r+1/2}}\,,\qquad \bar{\psi}_L^i(z)=\sum_s\frac{\bar{\psi}_s^i}{z^{s+1/2}}\,. \qquad \text{(B.30)}$$

Here

$$r\in\mathbb{Z}+\frac{1}{2}+\tilde{\phi}\,,\qquad s\in\mathbb{Z}+\frac{1}{2}-\tilde{\phi}\,, \qquad \text{(B.31)}$$

where $\tilde{\phi}\equiv\phi+\phi_0$. ϕ is equivalent to $\phi+1$ and we may therefor assume $-1/2\le\phi<1/2$. A similar expansion in $\bar{z}$ holds for ψ_R^i and $\bar{\psi}_R^i$, but with $\tilde{\phi}\to-\tilde{\phi}$ (due to the same reason as in the bosonic fields).

The anti-commutation relations are

$$\{\psi_r^i,\bar{\psi}_s^j\}=2\delta_{r+s}\,\eta^{ij}\,, \qquad \text{(B.32)}$$

and the Virasoro generators take the form

$$L_n=\frac{1}{4}\sum_r(2r-n):\bar{\psi}_{n-r}^i\psi_{i\,r}:+a\,\delta_n\,. \qquad \text{(B.33)}$$

The ground state $|0\rangle_\psi$ is defined by

$$\psi^i_r|0\rangle_\psi = \bar{\psi}^i_r|0\rangle_\psi = 0\,, \qquad \text{for } r > 0\,. \tag{B.34}$$

Note that for a half-integer $\tilde{\phi}$ there will be 2 different ground states for every complex fermion ψ^i, one annihilated by ψ^i_0 and the other by $\bar{\psi}^i_0$. These operators may also be used to move from one ground state to the other. Totally there are $2^D \cdot 2^D$ such states, for D holomorphic and D antiholomorphic complex fermions. All of them appear in the partition function.

We will calculate the value of a in the w coordinate separately for two cases:

case (i): $0 \leq \tilde{\phi} + \frac{1}{2} < 1$. Then for every holomorphic complex fermion:

$$a = -\frac{1}{2}\sum_{n=0,1,\ldots}\left(n + \frac{1}{2} + \tilde{\phi}\right) - \frac{1}{2}\sum_{n=1,2,\ldots}\left(n - \frac{1}{2} - \tilde{\phi}\right) = -\frac{1}{24} + \frac{1}{2}\tilde{\phi}^2\,. \tag{B.35}$$

case (ii): $1 \leq \tilde{\phi} + \frac{1}{2} < 2$. Then for every holomorphic complex fermion:

$$a = -\frac{1}{2}\sum_{n=-1,0,\ldots}\left(n + \frac{1}{2} + \tilde{\phi}\right) - \frac{1}{2}\sum_{n=2,3,\ldots}\left(n - \frac{1}{2} - \tilde{\phi}\right) = -\frac{1}{24} + \frac{1}{2}(\tilde{\phi} - 1)^2\,. \tag{B.36}$$

For the computation of the $N = 2$ string partition function, we take two holomorphic complex fermions and two antiholomorphic complex fermions. In the untwisted sector ($\phi_0 = 0$), one may use the case (i), which is consistent with $-1/2 \leq \phi < 1/2$. One gets

$$\begin{aligned}
\mathrm{tr}_\phi\left(q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) &= (q\bar{q})^{-\frac{1}{12}+\phi^2}\cdot\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}+\phi}e^{-2\pi i\theta})^2(1+q^{n-\frac{1}{2}-\phi}e^{2\pi i\theta})^2\times\\
&\quad\times(1+\bar{q}^{n-\frac{1}{2}-\phi}e^{-2\pi i\theta})^2(1+\bar{q}^{n-\frac{1}{2}+\phi}e^{2\pi i\theta})^2\\
&= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi\\-\theta\end{bmatrix}(0,\tau)\right|^4,\\
\mathrm{tr}_\phi\left(\mathbf{r}\cdot q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) &= (q\bar{q})^{-\frac{1}{12}+\phi^2}\cdot\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}+\phi}e^{-2\pi i\theta})^2(1-q^{n-\frac{1}{2}-\phi}e^{2\pi i\theta})^2\times\\
&\quad\times(1-\bar{q}^{n-\frac{1}{2}-\phi}e^{-2\pi i\theta})^2(1-\bar{q}^{n-\frac{1}{2}+\phi}e^{2\pi i\theta})^2\\
&= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi\\-\theta+1/2\end{bmatrix}(0,\tau)\right|^4.
\end{aligned} \tag{B.37}$$

Note that if $\phi = -1/2$, the different U(1)-charges of the fermionic zero-modes must be taken into account, but the result is the same.

In the twisted sector ($\phi_0 = 1/2$), the case (i) is applicable for $-1/2 \leq \phi < 0$, while the case (ii) for $0 \leq \phi < 1/2$. One can see that for $-1/2 \leq \phi < 0$,

$$\begin{aligned}
\mathrm{tr}_\phi\left(q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) &= (q\bar{q})^{-\frac{1}{12}+(\frac{1}{2}+\phi)^2}\cdot\prod_{n=1}^{\infty}(1+q^{n+\phi}e^{-2\pi i\theta})^2(1+q^{n-\phi-1}e^{2\pi i\theta})^2\times\\
&\quad\times(1+\bar{q}^{n-\phi-1}e^{-2\pi i\theta})^2(1+\bar{q}^{n+\phi}e^{2\pi i\theta})^2
\end{aligned}$$

$$= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi+1/2\\ -\theta\end{bmatrix}(0,\tau)\right|^4,$$

$$\begin{aligned}\mathrm{tr}_\phi\left(\mathbf{r}\cdot q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) &= (q\bar{q})^{-\frac{1}{12}+(\frac{1}{2}+\phi)^2}\cdot\prod_{n=1}^{\infty}(1-q^{n+\phi}e^{-2\pi i\theta})^2(1-q^{n-\phi-1}e^{2\pi i\theta})^2\times\\ &\quad\times(1-\bar{q}^{n-\phi-1}e^{-2\pi i\theta})^2(1-\bar{q}^{n+\phi}e^{2\pi i\theta})^2\\ &= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi+1/2\\ -\theta+1/2\end{bmatrix}(0,\tau)\right|^4.\end{aligned} \tag{B.38}$$

When $0 \leq \phi < 1/2$ we obtain

$$\begin{aligned}\mathrm{tr}_\phi\left(q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) &= (q\bar{q})^{-\frac{1}{12}+(\frac{1}{2}-\phi)^2}\cdot\prod_{n=1}^{\infty}(1+q^{n+\phi-1}e^{-2\pi i\theta})^2(1+q^{n-\phi}e^{2\pi i\theta})^2\times\\ &\quad\times(1+\bar{q}^{n-\phi}e^{-2\pi i\theta})^2(1+\bar{q}^{n+\phi-1}e^{2\pi i\theta})^2\\ &= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi-1/2\\ -\theta\end{bmatrix}(0,\tau)\right|^4\\ &= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi+1/2\\ -\theta\end{bmatrix}(0,\tau)\right|^4,\\ \mathrm{tr}_\phi\left(\mathbf{r}\cdot q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right) &= (q\bar{q})^{-\frac{1}{12}+(\frac{1}{2}-\phi)^2}\cdot\prod_{n=1}^{\infty}(1-q^{n+\phi-1}e^{-2\pi i\theta})^2(1-q^{n-\phi}e^{2\pi i\theta})^2\times\\ &\quad\times(1-\bar{q}^{n-\phi}e^{-2\pi i\theta})^2(1-\bar{q}^{n+\phi-1}e^{2\pi i\theta})^2\\ &= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi-1/2\\ -\theta+1/2\end{bmatrix}(0,\tau)\right|^4\\ &= \frac{1}{|\eta(\tau)|^4}\left|\vartheta\begin{bmatrix}\phi+1/2\\ -\theta+1/2\end{bmatrix}(0,\tau)\right|^4.\end{aligned} \tag{B.39}$$

Thus we get the same result as in (B.38).

Note that if $\phi = 0$, the different U(1)-charges of the fermionic zero-modes must be taken into account, but the result is the same.

B.2.3 Ghosts

The $N = 2$ fermionic ghosts consist of two sets of holomorphic and antiholomorphic ghosts. One is the usual conformal ghost - anti-ghost pair denoted as (b, c) with $\lambda = 2$. The other set is the ghost and anti-ghost pair $(\hat{b}, \hat{c})$, which is associated with the U(1) gauge symmetry and has $\lambda = 1$. The corresponding antiholomorphic sets are denoted $(\bar{b}, \bar{c})$ and $(\bar{\hat{b}}, \bar{\hat{c}})$. There are also two holomorphic sets of superghosts, related to the two generators of supersymmetry $G^{\pm}$; They are denoted $(\beta^{\pm}, \gamma^{\mp})$ (the sign indicates the U(1)-charge), and both have $\lambda = 3/2$. The corresponding antiholomorphic sets are denoted $(\bar{\beta}^{\pm}, \bar{\gamma}^{\mp})$.

We shall discuss the holomorphic (anti-)ghosts and (anti-)superghosts. A similar discussion applies to the antiholomorphic ones, but with $\phi \to -\phi$ (as in fermions). A general holomorphic ghost (or superghost) system (B, C) with $h(B) = \lambda,\ h(C) = 1 - \lambda$, obeying $BC = -\epsilon CB,\ \epsilon = \pm 1$, has the following stress tensor:

$$T(z) = \partial B\, C - \lambda\, \partial(BC)\,. \tag{B.40}$$

The mode expansions for (B,C) take the form

$$B(z)=\sum_r \frac{B_r}{z^{r+\lambda}}\,,\qquad C(z)=\sum_s \frac{C_s}{z^{s+1-\lambda}}\,. \tag{B.41}$$

Here

$$r\in\mathbb{Z}\,,\qquad s\in\mathbb{Z}\,, \tag{B.42}$$

for the (anti-)ghosts ($\epsilon=1$), and:

$$r\in\mathbb{Z}+\frac{1}{2}\pm\phi\,,\qquad s\in\mathbb{Z}+\frac{1}{2}\mp\phi\,, \tag{B.43}$$

for the (anti-)superghosts ($\epsilon=-1$), where the sign in front of ϕ is according to their U(1)-charge.

The anti-commutation (for $\epsilon=1$) or commutation (for $\epsilon=-1$) relations are given by

$$\{C_s,B_r\}=\delta_{r+s}\,,\qquad [C_s,B_r]=\delta_{r+s} \tag{B.44}$$

The Virasoro generators take the form

$$L_n=\sum_r (n\lambda-r):B_rC_{n-r}:+a\,\delta_n\,. \tag{B.45}$$

Here a is a zero point energy which depends on the definition of the ground state, namely a prescription of the operator ordering $:\ :$. The ground state $|0\rangle_{\rm gh}$ is defined as

$$B_r|0\rangle_{\rm gh}=0\,,\quad \text{for } r\geq 0\,,\qquad C_s|0\rangle_{\rm gh}=0\,,\quad \text{for } s>0\,. \tag{B.46}$$

The partition function does not include a sum over the fermionic ghosts zero-modes, because of the ghost and anti-ghost insertions in the torus amplitude (as in (B.20) for the U(1) ghosts insertions).

The constant a in the w coordinate reads:

1. For each set of holomorphic fermionic ghost and anti-ghost $\epsilon=+1$:

$$a=-\sum_{n=1,2,\ldots} n=\frac{1}{12}\,. \tag{B.47}$$

2. For each set of holomorphic superghost and anti-superghost $\epsilon=-1$:

$$a=\frac{1}{2}\sum_{n=0,1,\ldots}\left(n+\frac{1}{2}+\phi\right)+\frac{1}{2}\sum_{n=0,1,\ldots}\left(n+\frac{1}{2}-\phi\right)=\frac{1}{24}-\frac{\phi^2}{2}\,. \tag{B.48}$$

All the (anti-)ghosts and (anti-)superghosts are singlets under the orbifold group, so one obtains:

$$\begin{aligned}
Z_{\rm gh}&=\mathrm{tr}_\phi\left((-1)^{F_{\rm gh}}b_0c_0\bar{b}_0\bar{c}_0\,\hat{b}_0\hat{c}_0\bar{\hat{b}}_0\bar{\hat{c}}_0\,q^{L_0}\bar{q}^{\bar{L}_0}e^{-2\pi i\theta(J_0+\bar{J}_0)}\right)\\
&=\left|q^{\frac{1}{6}}\prod_{n=1}^{\infty}(1-q^n)^4\right|^2\cdot(q\bar{q})^{\frac{1}{12}-\phi^2}\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}+\phi}e^{-2\pi i\theta})^{-2}(1+q^{n-\frac{1}{2}-\phi}e^{2\pi i\theta})^{-2}\times\\
&\quad\times(1+\bar{q}^{n-\frac{1}{2}-\phi}e^{-2\pi i\theta})^{-2}(1+\bar{q}^{n-\frac{1}{2}+\phi}e^{2\pi i\theta})^{-2}\\
&=|\eta(\tau)|^{12}\left|\vartheta\begin{bmatrix}\phi\\-\theta\end{bmatrix}(0,\tau)\right|^{-4}.
\end{aligned} \tag{B.49}$$

Dependence on the U(1) charge. It is easy to see that in all the parts of the partition function calculated above, the ϕ dependence enters into L_0, $\tilde{L}_0$ through the form $J_0 \cdot \phi$ and $-\bar{J}_0 \cdot \phi$. Instead of tracing over oscillators with different ϕ's (and different conformal weights), we may sum only over oscillators with $\phi = 0$, but add $q^{J_0 \cdot \phi} \bar{q}^{-\bar{J}_0 \cdot \phi} = e^{2\pi i (J_0 \tau \phi + \bar{J}_0 \bar{\tau} \phi)}$ to the trace, and integrate over ϕ. Thus the full dependence on the U(1) charge is $e^{2\pi i [J_0(-\theta+\tau\phi) + \bar{J}_0(-\theta+\bar{\tau}\phi)]} = e^{2\pi i (J_0 u + \bar{J}_0 \bar{u})}$. This confirms (B.17).

C. Spectrum of the $N = 2$ orbifold

Different pictures. As in the $N = 1$ fermionic string, pictures play an important role in the $N = 2$ string. A picture is defined by the superghost sector ground state [14]. Three pictures will be mostly relevant to us.

- NS boundary conditions ($\phi = 0$): the ground states in the $(-1,-1)$, and $(0,0)$ pictures are defined by

$$\beta_r^{\pm}|0\rangle_{(-1,-1)} = 0\,, \quad \text{for } r \geq \frac{1}{2}\,, \qquad \gamma_s^{\pm}|0\rangle_{(-1,-1)} = 0\,, \quad \text{for } s \geq \frac{1}{2}\,, \tag{C.1}$$

$$\beta_r^{\pm}|0\rangle_{(0,0)} = 0\,, \quad \text{for } r \geq -\frac{1}{2}\,, \qquad \gamma_s^{\pm}|0\rangle_{(0,0)} = 0\,, \quad \text{for } s \geq \frac{3}{2}\,. \tag{C.2}$$

- R boundary conditions ($\phi = -1/2$): the ground state in the $(-1/2,-1/2)$ picture is defined by

$$\beta_r^{\pm}|0\rangle_{(-1/2,-1/2)} = 0\,, \quad \text{for } r \geq 0\,, \qquad \gamma_s^{\pm}|0\rangle_{(-1/2,-1/2)} = 0\,, \quad \text{for } s \geq 1\,. \tag{C.3}$$

Every state in the $(-1,-1)$ picture has a physically equivalent state in the $(0,0)$ picture, which is related to it by the picture changing operator. Its matter part is $G^{M\,+}_{-1/2} G^{M\,-}_{-1/2}$ [3, 5], where $G^{\pm}$ are the two spin $\frac{3}{2}$ generators of the $N = 2$ superconformal algebra. However, the converse is not true: not every state in the $(0,0)$ picture has a physically equivalent state in the $(-1,-1)$ picture.

C.1 BRST analysis: untwisted sector

We will consider only the massless states, since there are no massive states in the spectrum. The spectrum of the untwisted sector is found by taking $\mathbb{Z}_2$ invariant combinations of states of the unorbifolded theory. We begin by finding the spectrum of the latter. We will work in the holomorphic side. The zero point energy in the $(-1,-1)$, and $(-1/2,-1/2)$ pictures is zero. Thus, the massless states in these pictures are at level zero.

Acting with the BRST charge Q_{BRST} of the $N = 2$ strings (see [23] and the appendix B) on a level zero state $|\varphi\rangle$, produces a state which includes only ghosts zero modes c_0, $\hat{c}_0$, and in addition $\gamma_0^{\pm}$ in the $(-1/2,-1/2)$ picture. Also, every physical state is annihilated by the anti ghosts zero modes b_0, $\hat{b}_0$, and in addition $\beta_0^{\pm}$ in the $(-1/2,-1/2)$ picture. Therefor, for $|\varphi\rangle$ to be BRST invariant it is sufficient to require

$$\begin{aligned}
b_0 Q_{\text{BRST}}|\varphi\rangle &= \{b_0, Q_{\text{BRST}}\}|\varphi\rangle = L_0|\varphi\rangle = 0\,, \\
\hat{b}_0 Q_{\text{BRST}}|\varphi\rangle &= \{\hat{b}_0, Q_{\text{BRST}}\}|\varphi\rangle = J_0|\varphi\rangle = 0\,, \\
\beta_0^{\pm} Q_{\text{BRST}}|\varphi\rangle &= [\beta_0^{\pm}, Q_{\text{BRST}}]|\varphi\rangle = -G_0^{\pm}|\varphi\rangle = 0\,,
\end{aligned} \tag{C.4}$$

with the last equation being relevant only for the $(-1/2,-1/2)$ picture. In order for $|\varphi\rangle$ not to be BRST exact, it should not include ghost zero modes c_0, $\hat{c}_0$ and $\gamma_0^{\pm}$. Solving these conditions we get for the BRST cohomology:

- $(-1,-1)$ picture: the state $|0,k\rangle_{(-1,-1)}$ with $k^2=0$ is annihilated by L_0 and J_0. This is a massless scalar field [3].

- $(-1/2,-1/2)$ picture: define the ground state by $\bar{\psi}_0^i|0,k\rangle_{(-1/2,-1/2)}=0$, $i=1,2$. Then the state
$$\bar{k}_i\psi_0^i|0,k\rangle_{(-1/2,-1/2)}\,, \tag{C.5}$$
with $k^2=0$ is in the BRST cohomology. We will see later that this state is equivalent to the state $|0,k\rangle_{(-1,-1)}$.

- $(0,0)$ picture: here
$$L_0|0\rangle_{(0,0)}=\left(\frac{\alpha'}{4}k^2+\frac{1}{2}(\beta^+_{-1/2}\gamma^-_{1/2}+\beta^-_{-1/2}\gamma^+_{1/2})\right)|0\rangle_{(0,0)}=(\frac{\alpha'}{4}k^2-1)|0\rangle_{(0,0)}\,. \tag{C.6}$$
Therefor the massless states are at level one in this picture. $\gamma^{\pm}_{1/2}$ are now creation operators, and we can write any charge zero level one state $\mathcal{O}|0,k\rangle_{(0,0)}$ as
$$\mathcal{O}|0,k\rangle_{(0,0)}=\sum_n \mathcal{O}_{n,m}{\gamma^+_{1/2}}^n{\gamma^-_{1/2}}^m|0,k\rangle_{(0,0)}\,, \tag{C.7}$$
where $\mathcal{O}_{n,m}$ has charge $m-n$, conformal dimension $1+\frac{1}{2}(n+m)$, and does not include $\gamma^{\pm}_{1/2}$ oscillators. As we will show, we find the following physical states:

  1. $(\bar{k}_i\alpha^i_{-1}-k_i\bar{\alpha}^i_{-1}-(\bar{k}_i\psi^i_{-1/2})(k_i\bar{\psi}^i_{-1/2}))|0,k\rangle_{(0,0)}$, plus terms with ghosts oscillators. This is the massless scalar in the $(0,0)$ picture.
  2. $(\alpha^i_{-1}-\sqrt{2}\gamma^-_{1/2}b_{-1}\psi^i_{-1/2})|0,0\rangle_{(0,0)}$, $(\bar{\alpha}^i_{-1}-\sqrt{2}\gamma^+_{1/2}b_{-1}\bar{\psi}^i_{-1/2})|0,0\rangle_{(0,0)}$.[3] Combined with their anti holomorphic parts, these are global dilaton, B-field and metric degrees of freedom.

Let us see that. First, note that for any physical state of the form (C.7), when it is inserted in a scattering amplitude or an inner product, only $\mathcal{O}_{0,0}$ contributes. This is since the $\gamma^{\pm}_{1/2}$ oscillators annihilate the other parts of the scattering amplitude or inner product. For instance, the inner product of $\gamma^{\pm}_{1/2}|0,k\rangle_{(0,0)}$ with itself is zero.

If $\mathcal{O}|0,k\rangle_{(0,0)}$ is BRST invariant, it obeys
$$\begin{aligned}G^{\pm}_{-1/2}\mathcal{O}|0,k\rangle_{(0,0)} &= -[\beta^{\pm}_{-1/2},Q_{\rm BRST}]\mathcal{O}|0,k\rangle_{(0,0)}\\ &= Q_{\rm BRST}\beta^{\pm}_{-1/2}\mathcal{O}|0,k\rangle_{(0,0)}=-Q_{\rm BRST}\mathcal{O}_{\pm1}|0,k\rangle_{(0,0)}+\cdots\\ &= -\left(L^M_{-1}c_1+J^M_{-1}\hat{c}_1+G^{M\pm}_{-3/2}\gamma^{\mp}_{3/2}\right)\mathcal{O}_{\pm1}|0,k\rangle_{(0,0)}+\cdots\\ &= \left(L^M_{-1}A^{\pm}+J^M_{-1}B^{\pm}+G^{M\pm}_{-3/2}C\right)|0,k\rangle_{(0,0)}+\cdots\,,\end{aligned} \tag{C.8}$$

[3]For an earlier work on discrete states in $N=2$ strings on a flat background, see [16].

where dots refer to terms with ghost oscillators. $A^{\pm}$ and $B^{\pm}$ are operators of charge ± 1 and conformal dimension $1/2$, and independent of ghost oscillators. C is a number. $\mathcal{O}_{\pm 1}$ is a short notation for $\mathcal{O}_{0,1}$ and $\mathcal{O}_{1,0}$. The matter part of $G^{\pm}_{-1/2}\mathcal{O}|0,k\rangle_{(0,0)}$ is $G^{M\pm}_{-1/2}\mathcal{O}^M|0,k\rangle_{(0,0)}$ and thus

$$G^{M\pm}_{-1/2}\mathcal{O}^M|0,k\rangle_{(0,0)} = \left(L^M_{-1}A^{\pm} + J^M_{-1}B^{\pm} + G^{M\pm}_{-3/2}C\right)|0,k\rangle_{(0,0)}\,. \tag{C.9}$$

$\mathcal{O}^M|0,k\rangle_{(0,0)}$ is a level one state with charge zero, and no ghost oscillators. Therefor it is of the form

$$\mathcal{O}^M|0,k\rangle_{(0,0)} = \left(A_i\alpha^i_{-1} + A'_i\bar{\alpha}^i_{-1} + P_{ij}\psi^i_{-1/2}\bar{\psi}^j_{-1/2}\right)|0,k\rangle_{(0,0)}\,. \tag{C.10}$$

For $k \neq 0$, any solution of (C.9) is a linear combination of state (1) and the null states $\frac{1}{2}(\bar{k}_i\alpha^i_{-1} + k_i\bar{\alpha}^i_{-1})|0,k\rangle_{(0,0)} = L^M_{-1}|0,k\rangle_{(0,0)}$, which is the matter part of $Q_{\mathrm{BRST}}b_{-1}|0,k\rangle_{(0,0)}$, and $\frac{1}{2}\psi_{i\ -1/2}\bar{\psi}^i_{-1/2}|0,k\rangle_{(0,0)} = J^M_{-1}|0,k\rangle_{(0,0)}$, which is the matter part of $Q_{\mathrm{BRST}}\hat{b}_{-1}|0,k\rangle_{(0,0)}$. For $k = 0$, the solutions of (C.9) are linear combinations of (2) and the null state $J^M_{-1}|0,0\rangle_{(0,0)}$.

The untwisted sector of the theory on the orbifold consists of $\mathbb{Z}_2$-invariant combinations of states (1) and (2). Therefor the massless scalars appear in combinations of the form $(|0,k\rangle + |0,-k\rangle)$.

C.2 BRST analysis: twisted sector

The boundary conditions for the matter fields imply that ∂X^i, $\partial\bar{X}^i$ are expanded in half integer oscillators $\alpha^i_{n+1/2}$, $\bar{\alpha}^i_{n+1/2}$, and therefor there are no momentum degrees of freedom. ψ^i and $\bar{\psi}^i$ are expanded in half integer oscillators in the $(-1/2,-1/2)$ picture, and in integer ones in the $(-1,-1)$ and $(0,0)$ pictures. The same holds for the anti holomorphic side. L_0 and J_0 have different normal ordering constants relative to the untwisted sector.

- $(-1,-1)$ picture: the normal ordering constant of L_0 is $+1/2$, and condition (C.4) cannot be satisfied.
- $(-1/2,-1/2)$ picture: L_0 has zero normal ordering constant. The ground state is annihilated by J_0 and $G^{\pm}_0$ and is the only physical state.
- $(0,0)$ picture: the normal ordering constants of L_0 and J_0 are $-1/2$ and -1, respectively. Therefor the matter part of any physical state takes the form

  $$\left(C_{ij}\alpha^i_{-1/2}\psi^j_0 + D_{ij}\bar{\alpha}^i_{-1/2}\psi^j_0\right)|\sigma\rangle_{(0,0)}\,, \tag{C.11}$$

  where the twisted sector ground state $|\sigma\rangle_{(0,0)}$ is annihilated by $\bar{\psi}^i_0$, $i = 1,2$. Following the steps leading to (C.9), we get

  $$G^{M\pm}_{-1/2}\mathcal{O}^M|\sigma\rangle_{(0,0)} = \left(L^M_{-1}A^{\pm} + J^M_{-1}B^{\pm}\right)|\sigma\rangle_{(0,0)}\,, \tag{C.12}$$

  where $A^{\pm}$, $B^{\pm}$ have conformal dimension zero and charge two (for the plus sign), and zero (for the minus sign). Therefor C_{ij} is proportional to ϵ_{ij} and D_{ij} is proportional to η_{ij}, and we get two physical states, whose matter parts read

$$\sqrt{2}\epsilon_{ij}\alpha^i_{-1/2}\psi^j_0|\sigma\rangle_{(0,0)} = \frac{1}{\sqrt{2}}\alpha^i_{-1/2}\bar{\psi}_{i\ 0}\psi^1_0\psi^2_0|\sigma\rangle_{(0,0)} = G^{-\ M}_{-1/2}\psi^1_0\psi^2_0|\sigma\rangle_{(0,0)}\,,$$
$$\frac{1}{\sqrt{2}}\eta_{ij}\bar{\alpha}^i_{-1/2}\psi^j_0|\sigma\rangle_{(0,0)} = G^{+\ M}_{-1/2}|\sigma\rangle_{(0,0)}\,. \qquad (C.13)$$

These states have corresponding states in the $(-1,0)$ picture and in the $(0,-1)$ picture[4]

$$\psi^1_0\psi^2_0|\sigma\rangle_{(-1,0)}\,, \qquad |\sigma\rangle_{(0,-1)}\,. \qquad (C.14)$$

We will later see that these state are physically equivalent to the twisted ground state in the $(-1/2,-1/2)$ picture, and there is only one twisted state.

C.3 Vertex operators

Denote by σ^i and s^i, $i=1,2$, the bosonic and fermionic twist fields for the $\mathbb{Z}_2$ orbifold [17]. Their OPEs (omitting index i) read

$$\partial X(z)\sigma(w,\bar{w}) \sim (z-w)^{-1/2}\tau(w,\bar{w})\,, \qquad \psi(z)s(w) \sim (z-w)^{-1/2}t(w)\,,$$
$$\partial\bar{X}(z)\sigma(w,\bar{w}) \sim (z-w)^{-1/2}\tau'(w,\bar{w})\,, \qquad \bar{\psi}(z)s(w) \sim (z-w)^{1/2}t'(w)\,, \qquad (C.15)$$

where τ, τ' and t, t' are excited bosonic and fermionic twist fields, respectively. Bosonizing the fermionic fields

$$\psi^i = e^{iH_i}\,, \qquad \bar{\psi}_i = e^{-iH_i}\,, \qquad H_i(z)H_i(w) \sim -\log(z-w)\,, \qquad (C.16)$$

We obtain $s^i = e^{-iH_i/2}$. Consider next the construction of the vertex operators.

Untwisted sector. The ghost part of the vertex operator corresponding to a state in the (n,m) picture is

$$V_g^{(n,m)} = c\,e^{n\phi^+}\,e^{m\phi^-}\,(z)\,\bar{c}\,e^{n\bar{\phi}^+}\,e^{m\bar{\phi}^-}\,(\bar{z})\,, \qquad (C.17)$$

where $\beta^{\mp}\gamma^{\pm} = \partial\phi^{\pm}$.

- $(-1,-1)$ picture: the vertex operator corresponding to the scalar takes the form

$$V_g^{(-1,-1)}\cdot e^{i(k_i\bar{X}^i(z,\bar{z})+\bar{k}_iX^i(z,\bar{z}))}\,. \qquad (C.18)$$

- $(-1/2,-1/2)$ picture: the vertex operator corresponding to the scalar (C.5) takes the form

$$V_g^{(-1/2,-1/2)}\cdot\left(\bar{k}_i\psi^i\,e^{-iH_1/2}e^{-iH_2/2}\,(z)\cdot(\text{h.c.})\right)\cdot e^{i(k_i\bar{X}^i(z,\bar{z})+\bar{k}_iX^i(z,\bar{z}))}\,. \qquad (C.19)$$

[4]The ground states in these pictures satisfy (C.1) for one (β,γ) system, and (C.2) for the other (β,γ) system.

- $(-1,0)$ picture: The matter part of the scalar state in this picture is $G^{M+}_{-1/2}|0,k\rangle_{(-1,0)}$ and the corresponding vertex operator takes the form

$$V_g^{(-1,0)}\cdot\left(\bar{k}_i\psi^i\,(z)\cdot(\text{h.c.})\right)\cdot e^{i(k_i\bar{X}^i(z,\bar{z})+\bar{k}_iX^i(z,\bar{z}))}\,. \tag{C.20}$$

- $(0,0)$ picture: apart from the one corresponding to the massless scalar, we found additional states in the $(0,0)$ picture compared to the $(-1,-1)$ picture. These are global degrees of freedom of the form

$$c\bar{c}\cdot\Big(G_{ij}\partial X^i\bar{\partial}\bar{X}^j(z,\bar{z})+G'_{ij}\partial X^i\bar{\partial}X^j(z,\bar{z})+\widetilde{G}_{ij}\partial\bar{X}^i\bar{\partial}\bar{X}^j(z,\bar{z})+\widetilde{G}'_{ij}\partial\bar{X}^i\bar{\partial}\bar{X}^j(z,\bar{z})\Big)\,. \tag{C.21}$$

G, G', $\widetilde{G}$ and $\widetilde{G}'$ are number matrices.

Twisted sector.

- $(-1/2,-1/2)$ picture: the twisted state is represented by the vertex operator

$$V_g^{(-1/2,-1/2)}\cdot\,\sigma^1(z,\bar{z})\,\sigma^2(z,\bar{z})\,. \tag{C.22}$$

- $(-1,0)$ picture: the twisted state (C.14) is represented by the vertex operator

$$V_g^{(-1,0)}\cdot\,e^{iH_1/2}\,e^{iH_2/2}(z)\cdot(\text{h.c.})\cdot\,\sigma^1(z,\bar{z})\,\sigma^2(z,\bar{z})\,. \tag{C.23}$$

- $(0,-1)$ picture: the twisted state (C.14) is represented by the vertex operator

$$V_g^{(0,-1)}\cdot\,e^{-iH_1/2}\,e^{-iH_2/2}(z)\cdot(\text{h.c.})\cdot\,\sigma^1(z,\bar{z})\,\sigma^2(z,\bar{z})\,. \tag{C.24}$$

Spectral flow. Spectral flows [24] are automorphisms of the $N=2$ SCA. They are characterized by a single parameter η and map the $N=2$ generators as

$$\begin{aligned}
L_n^\eta &= L_n+\eta J_n+\frac{c}{6}\eta^2\,\delta_n\,,\\
J_n^\eta &= J_n+\frac{c}{3}\eta\,\delta_n\,,\\
G_r^{\eta\pm} &= G^\pm_{r\pm\eta}\,.
\end{aligned} \tag{C.25}$$

A spectral flow operator which implements these transformations takes the form

$$U_\eta=e^{-\eta(iH_1+iH_2+\phi^--\phi^+-\hat{b}c)}=e^{-\eta(iH_1+iH_2+\phi^--\phi^+)}(1+\eta\hat{b}c)\,. \tag{C.26}$$

Note that the U(1) current takes the form

$$J=\partial(iH_1+iH_2+\phi^--\phi^+-\hat{b}c)=\partial\Phi\,, \tag{C.27}$$

and $U_\eta=e^{-\eta\Phi}$.

U_η maps a state in the (n,m)-picture (C.17) to one in the $(n+\eta,m-\eta)$-picture. and η can be regarded as a Wilson line, turned on around the state. In particular, the spectral flows can map a state in the $(-1/2,-1/2)$ picture to a state in the $(-1,0)$ and $(0,-1)$ pictures, but not to a state in the $(-1,-1)$. For example, the state (C.22) gets mapped to (C.23) and (C.24) by the spectral flow with $\eta=\pm1/2$, and the state (C.19) to (C.20) by the spectral flow with $\eta=-1/2$.

Summary. The untwisted sector consists of $\mathbb{Z}_2$ invariant combinations of the states in the unorbifolded theory. These are a massless scalar, and global degrees of freedom which describe the dilaton, metric and two-form field. The twisted sector consists of one twisted state, described, for instance, by the twisted ground state in the $(-1/2,-1/2)$ picture.

D. Four twisted states scattering amplitude

D.1 Finiteness and invariance under crossing symmetry

By using

$$x = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\,, \qquad F(x) = \theta_3^2(\tau)\,, \qquad 1-x = \frac{\theta_4^4(\tau)}{\theta_3^4(\tau)}\,. \tag{D.1}$$

one finds that

$$\begin{aligned}
\mathcal{A}_{tttt} &= \int dx d\bar{x}\, \frac{|x(1-x)|^{-4/3}|\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)|^{-8/3}}{\tau_2^2(x,\bar{x})} V_\Lambda \times \\
&\quad \times \sum_{v_1\in\Lambda_c, v_2\in\Lambda_{c'}} e^{-\frac{\pi}{4\tau_2}[v_2\bar{v}_2-\tau_1(v_1\bar{v}_2+\bar{v}_1 v_2)+|\tau|^2 v_1\bar{v}_1]} \\
&= \int dx d\bar{x}\, \frac{|4x(1-x)|^{-4/3}}{|\eta(\tau)|^8\tau_2^2(x,\bar{x})} V_\Lambda \sum_{v_1\in\Lambda_c, v_2\in\Lambda_{c'}} e^{-\frac{\pi}{4\tau_2}[v_2\bar{v}_2-\tau_1(v_1\bar{v}_2+\bar{v}_1 v_2)+|\tau|^2 v_1\bar{v}_1]}\,.
\end{aligned} \tag{D.2}$$

With this, crossing symmetry is manifest: The transformations $\tau \to -1/\tau$ and $\tau \to \tau+1$ amount to $x \to 1-x$ and $x \to x/(x-1)$, respectively.

We will now show that (3.8) is finite. Because of crossing symmetry, it is enough to show that the integral converges at $x \to 0$. In this limit $F(x) \to 1$ and $\tau(x) \sim -\frac{i}{\pi}\ln(\frac{x}{16})$. For simplicity we assume $\epsilon_2 = 0$ and that both complex dimensions of the T^4 have the same radius R, with the sum over $v^i_{1,2}$, $\bar{v}^i_{1,2}$ $(i=1,2)$ replaced by a sum over the integers or half-integers $n^\mu_{1,2}$ $(\mu = 1\ldots 4)$, so that $v^i_{1,2} = 2R(n^1_{1,2}+in^2_{1,2}, n^3_{1,2}+in^4_{1,2})$, $n^\mu_1 \in \mathbb{Z}+\frac{1}{2}\epsilon^\mu_1$ and $n^\mu_2 \in \mathbb{Z}+\frac{1}{2}\epsilon^\mu_3$.

Taking $x = re^{i\varphi}$, the integral in (3.8) over a small disk of radius $r_0 < 1$ around $x=0$ is

$$\begin{aligned}
&\pi^2\int_0^{2\pi} d\varphi \int_0^{r_0} dr\, \frac{1}{r\ln^2(r/16)} \sum_{n^\mu_{1,2}\in\mathbb{Z}+\frac{1}{2}\epsilon^\mu_{1,3}} e^{\frac{(\pi R)^2}{\ln(r/16)}(n^\mu_2-n^\mu_1\varphi/\pi)^2+R^2\ln(r/16)(n^\mu_1)^2} = \\
&\qquad = \pi^2\int_0^{2\pi} d\varphi \int_0^{y_0} dy \sum_{n^\mu_{1,2}\in\mathbb{Z}+\frac{1}{2}\epsilon^\mu_{1,3}} e^{-(\pi R)^2 y(n^\mu_2-n^\mu_1\varphi/\pi)^2-\frac{R^2}{y}(n^\mu_1)^2} \\
&\qquad < \pi^2 \sum_{n^\mu_1\in\mathbb{Z}+\frac{1}{2}\epsilon^\mu_1} e^{-\frac{R^2}{y_0}(n^\mu_1)^2} \int_0^{2\pi} d\varphi \sum_{n^{\mu\,\prime}_2\in\mathbb{Z}+\frac{1}{2}\epsilon^\mu_3-n^\mu_1\varphi/\pi} \int_0^{y_0} dy\, e^{-(\pi R)^2 y(n^{\mu\,\prime}_2)^2} \\
&\qquad \leq 4\pi^2 \sum_{n^\mu_1\in\mathbb{Z}} e^{-\frac{R^2}{y_0}(n^\mu_1)^2} \int_0^{2\pi} d\varphi \sum_{n^{\mu\,\prime}_2\in\mathbb{Z}} \int_0^{y_0} dy\, e^{-(\pi R)^2 y(n^{\mu\,\prime}_2)^2} \\
&\qquad = 8\pi^3 \sum_{n^\mu_1\in\mathbb{Z}} e^{-\frac{R^2}{y_0}(n^\mu_1)^2} \left(y_0 + \sum_{n^{\mu\,\prime}_2\in\mathbb{Z}-\{0\}} \frac{1-e^{-(\pi R)^2 y_0 (n^{\mu\,\prime}_2)^2}}{(\pi R)^2(n^{\mu\,\prime}_2)^2} \right)
\end{aligned}$$

$$< \frac{8\pi}{R^2}\sum_{n_1^\mu\in\mathbb{Z}} e^{-\frac{R^2}{y_0}(n_1^\mu)^2}\cdot\left(\pi^2R^2y_0+\sum_{n_2^{\mu}{}'\in\mathbb{Z}-\{0\}}\frac{1}{(n_2^{\mu}{}')^2}\right)=\text{finite}\,. \tag{D.3}$$

We substituted $y=-\frac{1}{\ln(r/16)}$, and defined $y_0\equiv-\frac{1}{\ln(r_0/16)}$. In the third line we defined $n_2^{\mu}{}'\equiv n_2^\mu-n_1^\mu\varphi/\pi$, and in line four we replaced n_1^μ, $n_2^{\mu}{}'$ by their integer values.

Note that the result is finite for a $(4,0)$ signature. For a $(2,2)$ signature there will be infinite values of n_1^μ and $n_2^{\mu}{}'$ for which their square is zero. However, the divergence of the amplitude in the $(2,2)$ signature is the usual divergence of the path integral in Minkowski space, and the amplitude should be calculated by wick rotating it to the $(4,0)$ signature.

$\mathcal{A}_{tttt}$ is finite in the non-compact orbifold $\mathbb{C}^2/\mathbb{Z}_2$ as well. In the limit $x\to0$, $F(x)\to 1$ and $\tau_2\sim-\frac{1}{\pi}\ln(|x|/16)$. Therefor in (3.12) the leading singularity of the integrand around $x=0$ is $\frac{\pi^2}{|x|^2\ln^2(|x|/16)}$. The integral converges in this area, since $dxd\bar{x}/(|x|\ln|x|)^2\sim dr/r\ln^2r=d(-1/\ln r)$, where $r\equiv|x|$. Using crossing symmetry for exploring the other non-trivial limits of this integrand, we see that (3.12) totally converges.

D.2 Poles

We shall see which poles in $p_{L,R}$ exist in the amplitude $\mathcal{A}_{tttt}$ in its form (3.9).

In the limit $x\to0$,

$$F(x)=1+\frac{1}{4}x+\cdots\,,\qquad F(1-x)=-\frac{1}{\pi}\left(\ln\left(\frac{x}{16}\right)+\frac{1}{4}x\left[\ln\left(\frac{x}{16}\right)+2\right]+\cdots\right). \tag{D.4}$$

In particular $w(x)\sim x/16$ as $x\to0$. Every term of the sum in the integrand of (3.9) can be expanded around $x=0$ in powers of x and $\bar{x}$, so that the contribution of a small disk of radius a, $D(a)$, around $x=0$ is, up to the $e^{-2\pi i(f_{\epsilon_2}-f_{\epsilon_3})\cdot p}$ phase

$$\begin{aligned}\int_{D(a)}dxd\bar{x}x^{-1+(p+v/2)^2/2}\,\bar{x}^{-1+(p-v/2)^2/2}\cdot\sum_{n=0}^\infty A_nx^n\cdot\sum_{k=0}^\infty\bar{A}_k\bar{x}^k&=\\ =2\pi\int_0^a drr^{-1+p^2+v^2/4}\cdot r^{n+k}\sum_{n,k=0}^\infty A_n\bar{A}_k\delta_{n-k+p\cdot v}&\\ =2\pi\sum_{n=0}^\infty\frac{a^{(p+v/2)^2+2n}}{(p+v/2)^2+2n}A_n\bar{A}_{n+p\cdot v}=2\pi\sum_{k=0}^\infty\frac{a^{(p-v/2)^2+2k}}{(p-v/2)^2+2k}A_{k-p\cdot v}\bar{A}_k\,.&\end{aligned} \tag{D.5}$$

A_n, $\bar{A}_k$ are constants which depend only on $(p+v/2)^2$ and $(p-v/2)^2$, respectively. In the second line of (D.5) we changed variables to polar coordinates, and the δ-Kronecker arises from integrating over the angular coordinate.

Equation (D.5) is a sum over poles of the form $(p+v/2)^2+2n$ (or $(p-v/2)^2+2k$). If for every $n>0$ A_n is proportional to $(p+v/2)^2+2n$, which also means that for every $k>0$ $\bar{A}_k$ is proportional to $(p-v/2)^2+2k$, then the residues of the poles with $n\neq0$ or $k\neq0$ vanish. In order to see that this is indeed the case, it is sufficient to check that (D.5) has no such poles in the case $v=0$. We checked this explicitly for $0<n\le20$.[5]

[5]For example, for $n=1,2$ we get $A_1=2^{-2(p+v/2)^2-2}((p+v/2)^2+2)$ and $A_2=2^{-2(p+v/2)^2-7}(4(p+v/2)^2+13)\cdot((p+v/2)^2+4)$.

If (D.5) has poles only at $n = k = 0$, then these poles are at $(p+v/2)^2 = (p-v/2)^2 = 0$. Equivalently, the poles are at $p_L^2 = p_R^2 = 0$. We find that $A_0 = \bar{A}_0 = 4^{-(p+v/2)^2}$. Equation (D.5) takes the form

$$2\pi \frac{(a/16)^{p_L^2}}{p_L^2} \delta_{p_L^2 - p_R^2} + f(a, p_L^2, p_R^2) \tag{D.6}$$

where f has no poles in p_L^2, p_R^2.

E. Two twisted and two untwisted states amplitude

We start with

$$\begin{aligned} \mathcal{A}_{tt,p,k} = & \int dz d\bar{z} \, \langle \sigma_1(z_\infty)\sigma_2(z_\infty) \, \frac{1}{2} \left(e^{ik_{L,R}\cdot X^{L,R}(w,\bar{w})} + e^{-ik_{L,R}\cdot X^{L,R}(w,\bar{w})} \right) \times \\ & \times \frac{1}{2} \left(V^{(0,0)}(p_{L,R}; z, \bar{z}) + V^{(0,0)}(-p_{L,R}; z, \bar{z}) \right) \, \sigma_1(0)\sigma_2(0) \rangle \times \\ & \times | \langle e^{-\phi^+(z_\infty)/2} e^{-\phi^+(w)} e^{-\phi^+(0)/2} \rangle \, \langle e^{-\phi^-(z_\infty)/2} e^{-\phi^-(w)} e^{-\phi^-(0)/2} \rangle \, \langle c(z_\infty) c(w) c(0) \rangle |^2, \end{aligned} \tag{E.1}$$

where $k_{L,R} \cdot X^{L,R} \equiv k_L \cdot X^L + k_R \cdot X^R$.

Taking $z_\infty \to \infty$, the ghost part is $|z_\infty|$, canceling the matter part at this limit. It is easy to see that the fermionic fields in $V^{(0,0)}$ do not contribute to the amplitude, and so we are left with:

$$\begin{aligned} & \frac{1}{4} \int dz \, d\bar{z} |z_\infty| \langle \sigma_1(z_\infty)\sigma_2(z_\infty) \left(e^{ik_{L,R}\cdot X^{L,R}(w,\bar{w})} + e^{-ik_{L,R}\cdot X^{L,R}(w,\bar{w})} \right) (p_L \mathcal{J} \partial X)(p_R \mathcal{J} \bar{\partial} X) \times \\ & \times \left(e^{ip_{L,R}\cdot X^{L,R}(z,\bar{z})} + e^{-ip_{L,R}\cdot X^{L,R}(z,\bar{z})} \right) \sigma_1(0)\sigma_2(0) \rangle \, . \end{aligned} \tag{E.2}$$

We need to calculate the bosonic fields Green function in the presence of two twists. In order to simplify the notation, we will work in one complex dimension. The Green function is defined as

$$\begin{aligned} G(z, \bar{z}, w, \bar{w}) & \equiv \langle \sigma(\infty) \, X(z, \bar{z}) \, \bar{X}(w, \bar{w}) \, \sigma(0) \rangle \\ & = -2 \bigg(\int dz \, dw \, g(z, w) + \int d\bar{z} \, dw \, h(\bar{z}, w) + \int dz \, d\bar{w} \, \bar{h}(z, \bar{w}) + \int d\bar{z} \, d\bar{w} \, \bar{g}(\bar{z}, \bar{w}) \bigg), \\ g(z, w) & \equiv -\frac{1}{2} \langle \sigma(\infty) \, \partial_z X(z) \, \partial_w \bar{X}(w) \, \sigma(0) \rangle \, , \\ h(\bar{z}, w) & \equiv -\frac{1}{2} \langle \sigma(\infty) \, \bar{\partial}_{\bar{z}} X \, (\bar{z}) \partial_w \bar{X}(w) \, \sigma(0) \rangle \, . \end{aligned} \tag{E.3}$$

Note that $\langle \sigma(\infty) \, \sigma(0) \rangle \equiv \lim_{z_\infty \to \infty} |z_\infty|^{1/2} \langle \sigma(z_\infty) \sigma(0) \rangle = 1$.

Define

$$\begin{aligned} g(z, w; z_i, \bar{z}_i) & \equiv -\frac{1}{2} \frac{\langle \sigma(z_1, \bar{z}_1) \, \partial X(z) \, \partial \bar{X}(w) \, \sigma(z_2, \bar{z}_2) \rangle}{\langle \sigma(z_1, \bar{z}_1) \sigma(z_2, \bar{z}_2) \rangle} \, , \\ h(\bar{z}, w; z_i, \bar{z}_i) & \equiv -\frac{1}{2} \frac{\langle \sigma(z_1, \bar{z}_1) \, \bar{\partial} X(\bar{z}) \, \partial \bar{X}(w) \, \sigma(z_2, \bar{z}_2) \rangle}{\langle \sigma(z_1, \bar{z}_1) \sigma(z_2, \bar{z}_2) \rangle} \, , \end{aligned} \tag{E.4}$$

and $g(z,w) = g(z,w;\infty,0)$, $h(\bar z,w) = h(\bar z,w;\infty,0)$. Using the behavior of ∂X as $z,w\to z_i$, and the fact that g has a double pole with residue one as $z\to w$ (and no single pole), while h has no such poles, and using the transformations of g, h under the coordinate transformations $(z\to z+a)$, $(z\to\lambda\cdot z)$, we find that

$$\begin{aligned} g(z,w;z_i,\bar z_i) &= (z-z_1)^{-1/2}(z-z_2)^{-1/2}(w-z_1)^{-1/2}(w-z_2)^{-1/2}\times \\ &\quad\times\left[\frac{(z-z_1)(w-z_2)+(z-z_2)(w-z_1)}{2(z-w)^2}+A(z_i,\bar z_i)\right], \\ h(\bar z,w;z_i,\bar z_i) &= (\bar z-\bar z_1)^{-1/2}(\bar z-\bar z_2)^{-1/2}(w-z_1)^{-1/2}(w-z_2)^{-1/2}B(z_i,\bar z_i)\,. \end{aligned} \tag{E.5}$$

By using the transformations of g, h under the coordinate transformation $(z\to -1/z)$, we see that $A=B=0$. Setting $z_2=0$ and $z_1\to\infty$, we have

$$\begin{aligned} g(z,w) &= z^{-1/2}w^{-1/2}\frac{z+w}{2(z-w)^2}\,, \\ h(\bar z,w) &= 0 \end{aligned} \tag{E.6}$$

So we arrive at

$$G(z,\bar z,w,\bar w) = -2\left(\int dzdw\, g(z,w) + \int d\bar z d\bar w\, \bar g(\bar z,\bar w)\right) = -2\ln\left|\frac{w^{1/2}-z^{1/2}}{w^{1/2}+z^{1/2}}\right|^2 + C\,. \tag{E.7}$$

The constant C is zero. This can be seen by computing $G(z,\bar z,w,\bar w)$ using the state-operator correspondence

$$\begin{aligned} G(z,\bar z,w,\bar w) &= \langle\sigma|\, X(z,\bar z)\,\bar X(w,\bar w)\,|\sigma\rangle \\ &= -\sum_{m,n\in\mathbb{Z}}\langle\sigma|\,\alpha_{m-1/2}\frac{z^{-m+1/2}}{-m+1/2}\,\bar\alpha_{n-1/2}\frac{w^{-n+1/2}}{-n+1/2}\,|\sigma\rangle + \text{(c.c.)} \\ &= 2\sum_{m>0}\frac{z^{-m+1/2}w^{m-1/2}}{m-1/2} + \text{(c.c.)} = -2\ln\left|\frac{w^{1/2}-z^{1/2}}{w^{1/2}+z^{1/2}}\right|^2. \end{aligned} \tag{E.8}$$

Expression (E.8) splits into the holomorphic Green function X_LX_L and the antiholomorphic Green function X_RX_R. We denote them by $G(z,w)$ and $\bar G(\bar z,\bar w)$, respectively. The matter part of the two twisted two untwisted states scattering amplitude can then be calculated by the contractions of the bosonic matter fields using $G(z,w)$ and $\bar G(\bar z,\bar w)$ [17]. Recalling that we have two complex dimensions, we get

$$\begin{aligned} (z_\infty)^{1/2}\left\langle\sigma_1(z_\infty)\sigma_2(z_\infty)\, e^{ik_L\cdot X^L(w)}\, e^{ip_LX^L(z)}\,\sigma_1(0)\sigma_2(0)\right\rangle &= e^{-k_L\cdot p_L G(z,w)} \\ &= \left(\frac{w^{1/2}-z^{1/2}}{w^{1/2}+z^{1/2}}\right)^{2k_L\cdot p_L}, \end{aligned} \tag{E.9}$$

and similarly for the antiholomorphic side.

Every insertion of $\partial X^i_L(z)$, $\partial\bar X^i_L(z)$, $\bar\partial X^i_R(\bar z)$ or $\bar\partial\bar X^i_R(\bar z)$ results in multiplying the result of (E.9) by

$$ik^i_L\partial_z G(z,w) = -\frac{2ik^i_L w^{1/2}z^{-1/2}}{(z-w)}\,, \tag{E.10}$$

and its corresponding conjugates, respectively (due to its contraction with $e^{ik_{L,R}\cdot X^{L,R}(w)}$).

Note that the on-shell conditions $p_L^2 = k_L^2 = 0$ implies that $2k_L \cdot p_L = (k_L + p_L)^2 = -(k_L - p_L)^2$. Therefor the r.h.s. of (E.9) diverges in the limit $z \to (1 \pm i\epsilon)w$ for $(k_L \pm p_L)^2 < 0$. It is instructive to compare this with the unorbifolded OPE

$$e^{ik_L X^L(w)} e^{ip_L X^L(z)} \sim (w - z)^{(k_L+p_L)^2/2} e^{i(k_L+p_L)X^L}\,, \tag{E.11}$$

which diverges for $(k_L + p_L)^2 < 0$ in the limit $z \to w$, and represents the appearance of an intermediate state of (left) momentum $p_L + k_L$. On the orbifold a similar thing happens, but two intermediate states appear, rather than one. In the limit $z \to (1 + i\epsilon)w$, $X^L(z) \to X^L(w)$. Thus, the untwisted states in the l.h.s. of (E.9) combine to an intermediate state $e^{i(k_L+p_L)X^L}$, of left momentum $k_L + p_L$. In the limit $z \to (1 - i\epsilon)w$, $X^L(z) \to -X^L(w)$, and the untwisted states in the l.h.s. of (E.9) combine to an intermediate state $e^{i(k_L-p_L)X^L}$, of left momentum $k_L - p_L$. A similar discussion holds on the antiholomorphic side. Therefor, the untwisted states combine to form a state of left and right momentum $k_{L,R} + p_{L,R}$, and a state of left and right momentum $k_{L,R} - p_{L,R}$. This is to be expected, since every untwisted state of momentum $p_{L,R}$ includes also a piece carrying momentum $-p_{L,R}$, since its vertex operator is $\frac{1}{2}(e^{ip_{L,R}\cdot X^{L,R}} + e^{-ip_{L,R}\cdot X^{L,R}})$.

Fixing $w = 1$ we get

$$\begin{aligned}
\mathcal{A}_{tt,p,k} &= -\int dz\, d\bar z\, (p_L \mathcal{J} k_L)\,(p_R \mathcal{J} k_R) \left|\frac{z^{-1/2}}{z-1}\right|^2 \left(\frac{1 - z^{1/2}}{1 + z^{1/2}}\right)^{2k_L\cdot p_L} \left(\frac{1 - \bar z^{1/2}}{1 + \bar z^{1/2}}\right)^{2k_R\cdot p_R} + \\
&\quad + (p_{L,R} \leftrightarrow -p_{L,R}) + (k_{L,R} \leftrightarrow -k_{L,R}) + (p_{L,R}, k_{L,R} \leftrightarrow -p_{L,R}, -k_{L,R}) \\
&= -2\int dz d\bar z\, (p_L \mathcal{J} k_L)\,(p_R \mathcal{J} k_R)\, |z(z-1)^2|^{-1} \times \\
&\quad \times \left[\left(\frac{1 - z^{1/2}}{1 + z^{1/2}}\right)^{2k_L\cdot p_L} \left(\frac{1 - \bar z^{1/2}}{1 + \bar z^{1/2}}\right)^{2k_R\cdot p_R} + \left(\frac{1 - z^{1/2}}{1 + z^{1/2}}\right)^{-2k_L\cdot p_L} \left(\frac{1 - \bar z^{1/2}}{1 + \bar z^{1/2}}\right)^{-2k_R\cdot p_R}\right].
\end{aligned} \tag{E.12}$$

Denote

$$\mathcal{A}_{tt,p,k} = -2\,(p_L \mathcal{J} k_L)\,(p_R \mathcal{J} k_R) \Big[I\,(2k_L \cdot p_L, 2k_R \cdot p_R) + I\,(-2k_L \cdot p_L, -2k_R \cdot p_R) \Big]\,, \tag{E.13}$$

where

$$I(a_L, a_R) \equiv \int dz\, d\bar z\, |z(1-z)^2|^{-1} \left(\frac{1 - z^{1/2}}{1 + z^{1/2}}\right)^{a_L} \left(\frac{1 - \bar z^{1/2}}{1 + \bar z^{1/2}}\right)^{a_R}. \tag{E.14}$$

Then, substituting $z' = z^{1/2}$ (with z' in the upper complex plane), $v = \frac{1+z'}{2}$, we have

$$\begin{aligned}
I(a_L, a_R) &= 4\int_{\mathrm{Im}(z')\geq 0} dz'\, d\bar z'\, (1 - z')^{a_L-1}(1 + z')^{-a_L-1}(1 - \bar z')^{a_R-1}(1 + \bar z')^{-a_R-1} \\
&= \int_{\mathrm{Im}(v)\geq 0} dv\, d\bar v\, (1 - v)^{a_L-1} v^{-a_L-1} (1 - \bar v)^{a_R-1} \bar v^{-a_R-1}\,.
\end{aligned} \tag{E.15}$$

Taking $a_{L,R} \to -a_{L,R}$ is equivalent to the exchange $v \leftrightarrow 1 - v$. When $a_L - a_R$ is an integer we may write

$$\begin{aligned}
I(a_L, a_R) + I(-a_L, -a_R) &= \int_{\mathbb{C}} dv\, d\bar v\, (1 - v)^{a_L-1} v^{-a_L-1} (1 - \bar v)^{a_R-1} \bar v^{-a_R-1} \\
&= \pi \frac{\Gamma(-a_L)\Gamma(a_L)\Gamma(1)}{\Gamma(0)\Gamma(1 - a_R)\Gamma(1 + a_R)} = 0\,.
\end{aligned} \tag{E.16}$$

Since $k_L \cdot p_L - k_R \cdot p_R$ must be an integer for the theory to be modular invariant (as in any string theory with a toroidal compactification), we can use (E.16) to calculate the scattering amplitude (E.13). We thus get

$$\mathcal{A}_{tt,p,k} = 0\,. \tag{E.17}$$

All the above is easily generalized to the non-compact background $\mathbb{C}^2/\mathbb{Z}_2$ by setting $p_L = p_R$, $k_L = k_R$.

E.1 Study of $I(a_L, a_R)$

We may perform the integration in $I(2k_L \cdot p_L, 2k_R \cdot p_R)$ as follows. Taking $u = 1 - v$ and denoting $a_{L,R} \equiv 2k_{L,R} \cdot p_{L,R} = (k_{L,R} + p_{L,R})^2$, then for $a_L \neq a_R$,

$$\begin{aligned}
I(a_L, a_R) &= \int_{\mathrm{Im}(u)\leq 0} du\, d\bar{u}\, u^{a_L-1}(1-u)^{-a_L-1}\bar{u}^{a_R-1}(1-\bar{u})^{-a_R-1} \\
&= \int_{\mathrm{Im}(x)\leq 0} dx\, d\bar{x}\, x^{a_L-1}\bar{x}^{a_R-1} \\
&= \int_{\pi}^{2\pi} d\phi\, e^{i\phi(a_L-a_R)} \int_0^R dr\, r^{a_L+a_R-1} \\
&= -i\frac{1-e^{i\pi(a_L-a_R)}}{a_L-a_R}\int_0^R dr\, r^{a_L+a_R-1} = 0\,,
\end{aligned} \tag{E.18}$$

where we have substituted $x = \frac{u}{1-u}$, and later changed to polar coordinates. The result is zero because $a_L - a_R$ must be an even integer. We thus see that I has no contributions from $a_L \neq a_R$.

For $a_L = a_R \equiv a$

$$\begin{aligned}
I(a,a) &= \int_{\mathrm{Im}(u)\leq 0} du\, d\bar{u}\, |u|^{2a-2}|1-u|^{-2a-2} = \int_{\mathrm{Im}(x)\leq 0} dx\, d\bar{x}\, |x|^{2a-2} \\
&= \int_{|x|\leq 1, \mathrm{Im}(x)\leq 0} dx\, d\bar{x}\, |x|^{2a-2} + \int_{|x|\leq 1, \mathrm{Im}(x)\leq 0} dx\, d\bar{x}\, |x|^{-2a-2} \\
&= \pi\int_0^1 dr\, r^{2a-1} + \pi\int_0^1 dr\, r^{-2a-1} = \pi\frac{r^{2a}}{2a}\bigg|_0^1 + \pi\frac{r^{-2a}}{-2a}\bigg|_0^1
\end{aligned} \tag{E.19}$$

where we have substituted $x = \frac{u}{1-u}$ in the first line, $x \to -1/x$ in the second term of the second line, and $r = |x|$ in the last line.

The first term in the result of (E.19) is a pole at $a = 0$ with the branch cut at $a < 0$, coming from the limit $u \to 0$ (or $z \to 1 + i\epsilon$). $a = (p_L + k_L)^2 = (p_R + k_R)^2$, so this corresponds to an exchange of an untwisted state of left and right momenta $p_L + k_L, p_R + k_R$ with $(p_L + k_L)^2 = (p_R + k_R)^2 = 0$. The second term in the result of (E.19) is a pole at $a = 0$ with the branch cut at $a > 0$, coming from the limit $u \to 1$ (or $z \to 1 - i\epsilon$). $-a = (p_L - k_L)^2 = (p_R - k_R)^2$, so this corresponds to an exchange of an untwisted state of left and right momenta $p_L - k_L, p_R - k_R$ with $(p_L - k_L)^2 = (p_R - k_R)^2 = 0$. Identical poles arise from $I(-2k_L \cdot p_L, -2k_R \cdot p_R)$. Therefor only poles in $(p_{L,R} \pm k_{L,R})^2 = 0$ contribute to the scattering amplitude.

F. Twist fields in open strings

Consider open strings on the upper half plane, Im$z \geq 0$, with the boundary conditions $\partial X(z) = D\bar{\partial}X(\bar{z})$ ($D = \pm 1$) at $z = \bar{z}$. Close to the worldsheet boundary, (4.1) must be corrected by a boundary term. We may write:

$$\partial X(z)\sigma(w,\bar{w}) \sim f_1(z,w,\bar{w})\tau(w,\bar{w}), \qquad \bar{\partial}X(\bar{z})\sigma(w,\bar{w}) \sim f_2(\bar{z},w,\bar{w})\bar{\tau}(w,\bar{w}). \tag{F.1}$$

$f_{1,2}$ must be multiplied by -1 as z ($\bar{z}$) rotates around w ($\bar{w}$). Since $\sigma(w,\bar{w})$ corresponds to the unexcited twisted state we must have

$$f_1(z,w,\bar{w}) = (z-w)^{-1/2}\, h_1(z,w,\bar{w}), \qquad f_2(\bar{z},w,\bar{w}) = (\bar{z}-\bar{w})^{-1/2}\, h_2(\bar{z},w,\bar{w}), \tag{F.2}$$

such that

$$h_1(z,w,\bar{w})|_{z\to w} = \text{finite}, \qquad h_2(\bar{z},w,\bar{w})|_{\bar{z}\to\bar{w}} = \text{finite}. \tag{F.3}$$

The boundary conditions now imply

$$\begin{aligned} h_1(z,w,\bar{w}) &= (z-\bar{w})^{-1/2}\, \tilde{h}(w,\bar{w}), \\ h_2(\bar{z},w,\bar{w}) &= i(\bar{z}-w)^{-1/2}\, \tilde{h}(w,\bar{w}). \end{aligned} \tag{F.4}$$

and $\bar{\tau}(w,\bar{w}) = -iD\tau(w,\bar{w})$. One can take $\tilde{h}(w,\bar{w}) = (w-\bar{w})^{+1/2}$ so that the OPE reduces to that on S^2 as $z \to w$. We then have

$$\begin{aligned} \partial X(z)\sigma(w,\bar{w}) &\sim (z-w)^{-1/2}(z-\bar{w})^{-1/2}(w-\bar{w})^{+1/2}\,\tau(w,\bar{w}), \\ \bar{\partial}X(\bar{z})\sigma(w,\bar{w}) &\sim D(\bar{z}-\bar{w})^{-1/2}(\bar{z}-w)^{-1/2}(w-\bar{w})^{+1/2}\,\tau(w,\bar{w}). \end{aligned} \tag{F.5}$$

G. Two twist fields correlation function on D_2

For every complex dimension we may write

$$\begin{aligned} -\frac{1}{2}\frac{\langle \partial_z X(z)\partial_w \bar{X}(w)\sigma(z_1,\bar{z_1})\sigma(z_2,\bar{z_2})\rangle_{D_2}}{\langle \sigma(z_1,\bar{z_1})\sigma(z_2,\bar{z_2})\rangle_{D_2}} &= \\ = (z-z_1)^{-1/2}(z-z_2)^{-1/2}(z-\bar{z}_1)^{-1/2}(z-\bar{z}_2)^{-1/2} &\times \\ \times(w-z_1)^{-1/2}(w-z_2)^{-1/2}(w-\bar{z}_1)^{-1/2}(w-\bar{z}_2)^{-1/2} &\times g(z,w,z_i,\bar{z}_i), \end{aligned} \tag{G.1}$$

where the function $g(z,w,z_i,\bar{z}_i)$ is single-valued and is holomorphic in z and w. As $z \to w$ the correlation function must have a double pole with residue one, and no single pole. Thus, it should behave as $\sim 1/(z-w)^2$ plus finite terms. (G.1) must be invariant under worldsheet translations parallel to the worldsheet boundary (($z, \bar{z} \to z+a, \bar{z}+a$ for real a). After performing the coordinate transformations $z \to -1/z$, $w \to -1/w$ and $z \to \lambda z$, $w \to \lambda w$ (for real λ) (G.1) must be multiplied by $z^2 w^2$ and λ^{-2} respectively. Thus

$$\begin{aligned} g(z,w,z_i,\bar{z}_i) &= \\ = \frac{1}{6(z-w)^2}\Big[&(z-z_1)(z-z_2)(w-\bar{z}_1)(w-\bar{z}_2) + (z-z_1)(z-\bar{z}_1)(w-z_2)(w-\bar{z}_2) + \\ &+(z-z_1)(z-\bar{z}_2)(w-\bar{z}_1)(w-z_2) + (z-\bar{z}_1)(z-z_2)(w-z_1)(w-\bar{z}_2) + \\ &+(z-\bar{z}_1)(z-\bar{z}_2)(w-z_1)(w-z_2) + (z-z_2)(z-\bar{z}_2)(w-z_1)(w-\bar{z}_1)\Big] + \\ +A(z_1-z_2)(\bar{z}_1-\bar{z}_2) &+ B(z_1-\bar{z}_2)(\bar{z}_1-z_2), \end{aligned} \tag{G.2}$$

where A and B are constants. Subtracting $1/(z-w)^2$ and taking the limit $z \to w$ we get

$$\begin{aligned}
&\frac{\langle T(z)\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle}{\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle} = \\
&\quad = -\frac{1}{12}\Big[(z-\bar{z}_1)^{-1}(z-\bar{z}_2)^{-1}+(z-\bar{z}_1)^{-1}(z-z_2)^{-1}+(z-z_2)^{-1}(z-\bar{z}_2)^{-1}+ \\
&\qquad\qquad +(z-z_1)^{-1}(z-\bar{z}_1)^{-1}+(z-z_1)^{-1}(z-\bar{z}_2)^{-1}+(z-z_1)^{-1}(z-z_2)^{-1}\Big]+ \\
&\qquad +\frac{1}{8}\left[(z-z_1)^{-2}+(z-z_2)^{-2}+(z-\bar{z}_1)^{-2}+(z-\bar{z}_2)^{-2}\right]+ \\
&\qquad +(z-z_1)^{-1}(z-z_2)^{-1}(z-\bar{z}_1)^{-1}(z-\bar{z}_2)^{-1}\times \\
&\qquad \times\left(A(z_1-z_2)(\bar{z}_1-\bar{z}_2)+B(z_1-\bar{z}_2)(\bar{z}_1-z_2)\right). \qquad \text{(G.3)}
\end{aligned}$$

In the limit $z \to z_1$, the coefficient of $(z-z_1)^{-2}$ is the twist field conformal dimension 1/8, and the coefficient of the $(z-z_1)^{-1}$ term is $\partial_{z_1}\ln\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_{D_2}$. Therefor,

$$\begin{aligned}
\partial_{z_1}\ln\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_{D_2} &= -\frac{1}{12}\left((z_1-z_2)^{-1}+(z_1-\bar{z}_1)^{-1}+(z_1-\bar{z}_2)^{-1}\right)+ \\
&\quad +A\left(\frac{1}{z_1-\bar{z}_1}-\frac{1}{z_1-\bar{z}_2}\right)+B\left(\frac{1}{z_1-\bar{z}_1}-\frac{1}{z_1-z_2}\right). \qquad \text{(G.4)}
\end{aligned}$$

Repeating this procedure for the antiholomorphic part of the energy-momentum tensor and integrating, we get

$$\langle\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_{D_2} = (z_1-\bar{z}_1)^{A+B-\frac{1}{12}}(z_1-z_2)^{-B-\frac{1}{12}}(z_1-\bar{z}_2)^{-A-\frac{1}{12}}\times(\text{c.c.}). \qquad \text{(G.5)}$$

We have used the fact that the holomorphic and antiholomorphic parts must have the same constants A, B so that $T(z)=\bar{T}(\bar{z})$ on the worldsheet boundary.

References

[1] M. Ademollo et al., *Dual string with* U(1) *color symmetry*, *Nucl. Phys.* **B 111** (1976) 77.

[2] S.J. Gates Jr., L. Lu and R.N. Oerter, *Simplified* SU(2) *spinning string superspace supergravity*, *Phys. Lett.* **B 218** (1989) 33.

[3] H. Ooguri and C. Vafa, *Geometry of $N=2$ strings*, *Nucl. Phys.* **B 361** (1991) 469.

[4] Y.-K.E. Cheung, Y. Oz and Z. Yin, *Families of $N=2$ strings*, *JHEP* **11** (2003) 026 [hep-th/0211147].

[5] D. Glück, Y. Oz and T. Sakai, *The effective action and geometry of closed $N=2$ strings*, *JHEP* **07** (2003) 007 [hep-th/0304103].

[6] C.M. Hull, *The geometry of $N=2$ strings with torsion*, *Phys. Lett.* **B 387** (1996) 497 [hep-th/9606190].

[7] D. Glück, Y. Oz and T. Sakai, *D-branes in $N=2$ strings*, *JHEP* **08** (2003) 055 [hep-th/0306112].

[8] N. Ohta and S. Osabe, *Hidden extended superconformal symmetries in superstrings*, *Phys. Rev.* **D 39** (1989) 1641.

[9] N. Berkovits and C. Vafa, $N = 4$ *topological strings*, *Nucl. Phys.* **B 433** (1995) 123 [hep-th/9407190].

[10] O. Aharony, B. Fiol, D. Kutasov and D.A. Sahakyan, *Little string theory and heterotic/type-II duality*, *Nucl. Phys.* **B 679** (2004) 3 [hep-th/0310197].

[11] J. Bischoff and O. Lechtenfeld, *Path-integral quantization of the (2,2) string*, *Int. J. Mod. Phys.* **A 12** (1997) 4933 [hep-th/9612218].

[12] S.D. Mathur and S. Mukhi, *BRST quantization of twisted extended fermionic strings*, *Phys. Rev.* **D 36** (1987) 465.

[13] S.V. Ketov, O. Lechtenfeld and A.J. Parkes, *Twisting the* $N = 2$ *string*, *Phys. Rev.* **D 51** (1995) 2872 [hep-th/9312150].

[14] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.

[15] K. Junemann and O. Lechtenfeld, *Chiral brst cohomology of* $N = 2$ *strings at arbitrary ghost and picture number*, *Commun. Math. Phys.* **203** (1999) 53 [hep-th/9712182].

[16] J. Bienkowska, *The generalized no ghost theorem for* $N = 2$ *SUSY critical strings*, *Phys. Lett.* **B 281** (1992) 59 [hep-th/9111047].

[17] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The conformal field theory of orbifolds*, *Nucl. Phys.* **B 282** (1987) 13.

[18] K. Junemann and B. Spendig, *D-brane scattering of* $N = 2$ *strings*, *Phys. Lett.* **B 520** (2001) 163 [hep-th/0108069].

[19] N. Marcus, *The* $N = 2$ *open string*, *Nucl. Phys.* **B 387** (1992) 263 [hep-th/9207024].

[20] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[21] H. Ooguri and C. Vafa, *All loop* $N = 2$ *string amplitudes*, *Nucl. Phys.* **B 451** (1995) 121 [hep-th/9505183].

[22] E.S. Fradkin and A.A. Tseytlin, *Quantization of two-dimensional supergravity and critical dimensions for string models*, *Phys. Lett.* **B 106** (1981) 063.

[23] M. Li, *Gauge symmetries and amplitudes in* $N = 2$ *strings*, *Nucl. Phys.* **B 395** (1993) 129 [hep-th/9204027].

[24] A. Schwimmer and N. Seiberg, *Comments on the* $N = 2$, $N = 3$, $N = 4$ *superconformal algebras in two-dimensions*, *Phys. Lett.* **B 184** (1987) 191.